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SECONDARY COHOMOLOGY OPERATIONS: TWO FORMULAS.*[1]

By F. P. Peterson and N. Stein.

Introduction. In the past decade, there have been many important applications of algebraic topology involving such primary cohomology operations as cup products and the Steenrod reduced powers. Recently, it has become clear that many more problems can be attacked by considering secondary and higher order cohomology operations. Unfortunately, these are more difficult to compute than primary operations. In this paper, we give two formulas relating secondary cohomology operations to primary and functional primary cohomology operations, thus reducing the problem of computation to somewhat simpler problems.

In Chapter one, we discuss higher order cohomology operations, paying particular attention to the difficulties which arise in the non-stable cases. For example, in such cases the values of a secondary cohomology operation may be cosets of subgroups which depend on the variable to which the operation is applied and not only on the operation itself. In some cases they may be subsets of a group which are not necessarily cosets.

In Chapter two, we prove the following two formulas (stated here in the stable form). Let Φ be a secondary operation coming from the relation $\theta'\theta = 0$, where $\theta \in H^{n'+1}(\pi, n; \pi')$ and $\theta' \in H^{q+1}(\pi', n'+1; G)$. 1) Let $f: L \to K$ and let $u \in H^n(K; \pi)$ be such that $f^*(u) = 0$ and $\theta(u) = 0$. Then $f^*\Phi(u) = {}^1\theta'(\theta_f(u)) \in H^q(L; G)/{}^1\theta' f^*(H^{n'}(K; \pi'))$, where ${}^1\theta'$ denotes the suspension of θ'. 2) Let $f: L \to K$ and let $u \in H^n(K; \pi)$ be such that $f^*\theta(u) = 0$. Then $\Phi(f^*(u)) = \theta'_f(\theta(u)) \in H^q(L; G)/{}^1\theta'(H^{n'}(L; \pi')) + f^*(H^q(K; G))$. We also prove analogs of these formulas for some non-stable operations and for operations on more than one variable. There are similar formulas for higher order cohomology operations, but we shall not discuss them here.

In Chapter three, we give examples of applications of these two formulas to finding relations on secondary cohomology operations; e.g. we show that

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$Sq^2\Psi_n = 0$ for n odd, where Ψ_n is the secondary operation needed to classify maps into complex projective spaces (see Stein's thesis [13]). Finally, we show how the knowledge of secondary cohomology operations in the base of a fibre space gives information on primary cohomology operatins in the total space.

Throughout this paper, all statements refer to the total singular complexes of the spaces involved, and all homotopies are singular homotopies [16]. For simplicity, we will always use the geometric language.

A preliminary account of these results was given in [8].

Chapter I. Cohomology Operations.

1. Fibre spaces and principal fibre spaces. We will assume throughout this paper that the spaces are simply connected.

For any two spaces X and Y with base points $x \in X$, $y \in Y$, we denote by $\pi(X;Y)$ the set of homotopy classes of maps $(X,x) \to (Y,y)$. In many cases, e.g. whenever Y is a space of loops, $\pi(X;Y)$ has a natural group structure. In any case it contains a neutral element—the homotopy class of the constant map. We denote by ${}^1X$ the space of loops in X based at x and define ${}^rX = {}^1({}^{r-1}X)$. Finally, we denote by SX the reduced suspension of X and define $S^rX = S(S^{r-1}X)$.

Let $p: E \to B$ be a fibre space in the sense of Serre; let $b \in B$ and $e \in F = p^{-1}(b)$ be base points and let $i: F \to E$ be the inclusion map. Then according to Lemma 2.1 of [7], we have, for any space X, an exact sequence

$$(1.1) \qquad \cdots \to \pi(X;{}^rF) \xrightarrow{({}^ri)_\#} \pi(X;{}^rE) \xrightarrow{({}^rp)_\#} \pi(X;{}^rB) \to \pi(X;{}^{r-1}F) \to$$

$$\cdots \to \pi(X;F) \xrightarrow{i_\#} \pi(X;E) \xrightarrow{p_\#} \pi(X;B).$$

We now recall the notion of principal fibre space [9].

Definition. Let $p: E \to B$, b, e, and F be as above. We assume that F is a monoid (i.e. a topological semi-group with identity), and that there is given a map

$$\mu: F \times E \to E$$

such that the following diagrams are commutative (m is the multiplication in F and π is the projection onto the second factor):

$$\begin{array}{ccc} F \times F & \xrightarrow{m} & F \\ {\scriptstyle 1\times i}\downarrow & & \downarrow{\scriptstyle i} \\ F \times E & \xrightarrow{\mu} & E \end{array}$$

$$\begin{array}{ccc} F \times E & \xrightarrow{\mu} & E \\ {\scriptstyle 1\times p}\downarrow & & \downarrow{\scriptstyle p} \\ F \times B & \xrightarrow{\pi} & B. \end{array}$$

Let $E^* = \{(e_1, e_2) \in E \times E \mid p(e_1) = p(e_2)\}$ and defines maps

$$p_i \colon E^* \to E, \qquad i = 1, 2,$$

by setting $p_i(e_1, e_2) = e_i$. We assume that there is given a map

$$h \colon E^* \to F$$

such that the following diagram is commutative up to homotopy:

$$\begin{tikzcd} E^* \arrow[r, "{(h, p_1)}"] \arrow[dr, "p_2"'] & F \times E \arrow[d, "\mu"] \\ & E \end{tikzcd}$$

In this case, we say that we have a *principal fibre space.*

As was remarked in [9], the Moore path space is a principal fibre space and any fibre space which is induced from a principal fibre space by a map, is itself a principal fibre space.

It is easy to see that if F admits a classifying space $\chi(F)$, i.e. if there is a principal fiber space with fibre F, base space $\chi(F)$, and acyclic total space, from which any principal fibre space with fibre F is induced by a continuous map χ of its base into $\chi(F)$, then the exact sequence (1.1) can be extended by one term so as to end with $\xrightarrow{p_\#} \pi(X; B) \xrightarrow{\chi_\#} \pi(X; \chi(F))$. In particular, this holds whenever F is a loop-space.

We recall for later use Lemma 4.1 of [9].

LEMMA 1.2. *Let (E, p, B, F, μ, h) be a principal fibre space and let X be any space. Let $v, v' \in \pi(X; E)$. Then $p_\#(v) = p_\#(v')$ if and only if there exists $w \in \pi(X; F)$ such that $\mu_\#(w, v) = v'$.* [Here we use

$$\pi(X;F) \times \pi(X;E) = \pi(X;F \times E) \xrightarrow{\mu_\#} \pi(X;E).]$$

Also for later use we point out an obvious lemma.

LEMMA 1.3. *Under the same hypotheses, we have a commutative diagram*

$$\begin{tikzcd}
\pi(X;F\times E)\times H^i(E;G) \arrow[r, "\mu_\# \times 1"] \arrow[d, "1\times\mu^*"'] & \pi(X;E)\times H^i(E;G) \arrow[d] \\
\pi(X;F\times E)\times H^i(F\times E;G) \arrow[r] & H^i(X;G)
\end{tikzcd}$$

where the unlabelled arrows indicate operation of the first factor in the direct product on the second.

We note that it can be seen easily that the Moore path space is a homotopy associative principal fibre space in the sense that the following diagram is commutative up to homotopy:

$$\begin{tikzcd}
F\times F\times E \arrow[r, "1\times\mu"] \arrow[d, "m\times 1"'] & F\times E \arrow[d, "\mu"] \\
F\times E \arrow[r, "\mu"] & E.
\end{tikzcd}$$

Furthermore, a principal fibre space induced from a homotopy associative one is clearly homotopy associative. Finally, in a homotopy associative principal fibre space the following diagram is easily seen to be commutative up to homotopy (f is the identity of F, π denotes projection onto the second factor, and j is the inclusion):

$$\begin{tikzcd}
f\times E \arrow[r, "j\times 1"] \arrow[dr, "\pi"'] & F\times E \arrow[d, "\mu"] \\
 & E.
\end{tikzcd}$$

2. Postnikov spaces and their fibrations. The Postnikov scheme for determining the homotopy type of complexes [10] is based on the study of spaces with a finite number of non-trivial homotopy groups. The homotopy type of such a space is not uniquely determined by these groups but depends also on certain auxiliary invariants, known as k-invariants or Postnikov invariants. We will describe an inductive procedure for constructing the

Postnikov spaces which exhibits them as homotopy associative principal fibre spaces and in which the dependence on the k-invariants is indicated.

We recall first that for the case of a space with only one non-trivial homotopy group, i. e. an Eilenberg-MacLane space, the homotopy type is determined by the group. We assume that we are given a Postnikov space $\mathfrak{P}$ and we wish to construct a new space $\mathfrak{P}'$ which has the same homotopy groups as $\mathfrak{P}$ except in one dimension n which is assumed to be larger than the dimension of any non-trivial homotopy group of $\mathfrak{P}$, and that $\pi_n(\mathfrak{P}') \approx \pi$. For this, we consider the space $K(\pi, n+1)$ and the space E of Moore paths in $K(\pi, n+1)$ with fixed initial point, which according to § 1 is a principal fibre space over $K(\pi, n+1)$ with fibre the space of loops in $K(\pi, n+1)$, i. e. $K(\pi, n)$. For each map of $\mathfrak{P}$ into $K(\pi, n+1)$ we get an induced principal fibre space $\mathfrak{P}'$ with base $\mathfrak{P}$ and fibre $K(\pi, n)$.

$$\begin{tikzcd} K(\pi,n) \arrow[r] & \mathfrak{P}' \arrow[r] \arrow[d] & E \arrow[d] & K(\pi,n) \arrow[l] \\ & \mathfrak{P} \arrow[r] & K(\pi,n+1). & \end{tikzcd}$$

The homotopy sequence of this fibre space shows that $\mathfrak{P}'$ has the right homotopy groups. Altering the map $\mathfrak{P} \rightarrow K(\pi, n+1)$ within its homotopy class does not change the homotopy type of $\mathfrak{P}'$. According to the Hopf-Whitney theorem, the homotopy classes of maps $\mathfrak{P} \rightarrow K(\pi, n+1)$ are in natural one-one correspondence with the elements of the group $H^{n+1}(\mathfrak{P}; \pi)$. The element of this group corersponding to the map $\mathfrak{P} \rightarrow K(\pi, n+1)$ is the new k-invariant.

We remark that it is clear from this construction that the homotopy type of $\mathfrak{P}'$ is determined by the Postnikov system, i. e. the sequence of homotopy groups and k-invariants. However, it is unfortunately over-determined by this information in the sense that different Postnikov systems may correspond to the same homotopy type. Of course the homotopy groups cannot be altered beyond isomorphism but it is still an open question to say how much the k-invariants can be changed without affecting the homotopy type.

We will use the notation $\mathfrak{P}(\pi, n; \pi', n', \theta; \cdots; \pi^{(q)}, n^{(q)}, \theta^{(q-1)})$ for a space with homotopy groups $\pi^{(i)}$ in dimension $n^{(i)}$ and k-invariants

$$\theta^{(i-1)} \in H^{n^{(i)}+1}(\mathfrak{P}(\pi, n; \cdots; \pi^{(i-1)}, n^{(i-1)}, \theta^{(i-2)}); \pi^{(i)}).$$

3. Cohomology operations. Serre has remarked in [11] that due to the Hopf-Whitney theorem we can identify a universally defined cohomology operation with an element of a cohomology group of an Eilenberg-MacLane space. Motivated by this result, we give here a treatment of general cohomology operations. Our treatment gives a procedure for constructing these operations but we feel that it would be of interest to give an axiomatic characterization of them. J. F. Adams has recently given such a characterization for stable secondary operations.

We describe briefly the treatment of primary cohomology operations. Such an operation is uniquely determined by a cohomology class in an Eilenberg-MacLane space as follows: Let $\phi \in H^q(\pi, n; G)$. Let X be a space and let $h \in H^n(X; \pi)$. The homotopy classes of maps of X into $K(\pi, n)$ are in one-one correspondence with the elements of $H^n(X; \pi)$, and the correspondence is given by $f \leftrightarrow f^*(\iota)$, where $\iota \in H^n(\pi, n; \pi)$ is the basic class. Let $f\colon X \to K(\pi, n)$ be a map which corresponds to h, i.e. such that $f^*(\iota) = h$. Then we define $\phi(h) = f^*(\phi)$. Thus for any space X, ϕ is a natural function from $H^n(X; \pi)$ to $H^q(X; G)$.

For clarity, we will now describe the case of secondary operations and then show how to treat the general case. Let $\mathfrak{P} = \mathfrak{P}(\pi, n; \pi', n', \theta)$ be a Postnikov space, and let $\phi \in H^q(\mathfrak{P}; G)$. According to § 2, $\mathfrak{P}$ is a principal fibre space over $K(\pi, n)$ with fibre $K(\pi', n')$ and $K(\pi', n'+1)$ is a classifying space for this fibre space. For any space X, we have the exact sequence (1.1) with the extra term described at the end of § 1. We rewrite this sequence identifying $\pi(X; K(\pi, n))$ with $H^n(X; \pi)$:

$$(3.1) \qquad \cdots \to \pi(X; {}^1\mathfrak{P}) \xrightarrow{({}^1p)_\#} H^{n-1}(X; \pi) \xrightarrow{{}^1\theta} H^{n'}(X; \pi')$$

$$\xrightarrow{i_\#} \pi(X; \mathfrak{P}) \xrightarrow{p_\#} H^n(X; \pi) \xrightarrow{\theta} H^{n'+1}(X; \pi').$$

Let $h \in H^n(X; \pi)$ be such that $\theta(h) = 0$. Then by exactness of (3.1) there is a $v \in \pi(X; \mathfrak{P})$ such that $p_\#(v) = h$. Define $\Phi_v(h) = v^*(\phi) \in H^q(X; G)$ and $\Phi(h) = \{\Phi_v(h)\}$, i.e. $\Phi(h)$ is the collection of $\Phi_v(h)$ for all v such that $p_\#(v) = h$. Φ is the secondary operation determined by ϕ.

To define an $m+1$-ary operation now, let $\mathfrak{P} = \mathfrak{P}(\pi, n; \pi', n', \theta)$ let $\mathfrak{P}' = \mathfrak{P}(\pi, n; \pi', n', \theta; \pi'', n'', \theta')$, and in general, let

$$\mathfrak{P}^{(i)} = \mathfrak{P}(\pi, n; \pi', n', \theta; \cdots; \pi^{(i+1)}, n^{(i+1)}, \theta^{(i)}),$$

and let $\phi \in H^q(\mathfrak{P}^{(m-1)}; G)$. We have the exact couple described in [8], part of which is:

$$\begin{tikzcd}
H^{n^{(m)}}(X;\pi^{(m)}) \arrow[r, "i_{\#}^{(m-1)}"] & \pi(X;\mathfrak{P}^{(m-1)}) \arrow[r, "\phi_{\#}"] \arrow[d, "p_{\#}^{(m-1)}"'] & H^{q}(X;G) \\
H^{n^{(m-1)}}(X;\pi^{(m-1)}) \arrow[r, "i_{\#}^{(m-2)}"] & \pi(X;\mathfrak{P}^{(m-2)}) \arrow[r, "\theta_{\#}^{(m-1)}"] \arrow[d, "p_{\#}^{(m-2)}"'] & H^{n^{(m)}+1}(X;\pi^{(m)}) \\
H^{n^{(m-2)}}(X;\pi^{(m-2)}) \arrow[r, "i_{\#}^{(m-3)}"] & \pi(X;\mathfrak{P}^{(m-3)}) \arrow[r, "\theta_{\#}^{(m-2)}"] \arrow[d, "p_{\#}^{(m-3)}"'] & H^{n^{(m-1)}+1}(X;\pi^{(m-1)}) \\
 & \vdots \arrow[d, "p_{\#}'"'] & \\
H^{n'}(X;\pi') \arrow[r, "i_{\#}"] & \pi(X;\mathfrak{P}) \arrow[r, "\theta_{\#}^{(1)}"] \arrow[d, "p_{\#}"'] & H^{n''+1}(X;\pi'') \\
 & H^{n}(X;\pi) \arrow[r, "\theta"] & H^{n'+1}(X;\pi').
\end{tikzcd}$$

We recall that in this diagram, the sequences which begin at the far left, go one step to the right, one step down, and one step to the right, are exact. Furthermore, we note that if $\Theta^{(1)}$ is the secondary operation associated with $\theta^{(1)}$, then for any $h \in H^n(X;\pi)$ such that $\theta(h) = 0$, $\Theta^{(1)}(h)$ is the collection $\{\theta_{\#}^{(1)}(v)\}$ for all $v \in \pi(X;\mathfrak{P})$ such that $p_{\#}(v) = h$. We assume inductively that i-ary operations have been defined for $1 \leqq i \leqq m$ and we denote by $\Theta^{(i)}$ the $(i+1)$-ary operation associated with $\theta^{(i)}$. We assume that $\Theta^{(i)}$ is defined for those elements $h \in H^n(X;\pi)$ such that $0 \in \Theta^{(i-1)}(h)$ and that in this case, $\Theta^{(i)}(h)$ is the collection $\{\theta_{\#}^{(i)}(v)\}$ for all $v \in \pi(X;\mathfrak{P}^{(i-1)})$ such that $p_{\#}\,p_{\#}' \cdots p_{\#}^{(i-1)}(v) = h$. There are such elements v because of the assumption that $0 \in \Theta^{(i-1)}(h)$. We now assume that $0 \in \Theta^{(m-1)}(h)$—note that this implies that $\Theta^{(i)}(h)$ is defined for $0 \leqq i \leqq m-1$—and define $\Phi(h) \subset H^q(X;G)$ to be the set of elements $\{\phi_{\#}(v)\}$ for all $v \in \pi(X;\mathfrak{P}^{(m-1)})$ such that

$$p_{\#}p_{\#}' \cdots p_{\#}^{(m-2)}p_{\#}^{(m-1)}(v) = h.$$

Returning to the case of secondary operations, let $\mathfrak{P} = \mathfrak{P}(\pi, n; \pi', n', \theta)$, let $\phi \in H^q(\mathfrak{P};G)$ and let $\theta' = i^*(\phi) \in H^q(\pi', n'; G)$. We give a lemma now which will be used later:

LEMMA 3.2. *If $q < n + n'$, there is a unique element*

$$^{-1}\theta' \in H^{q+1}(\pi', n'+1; G)$$

such that $^{1}(^{-1}\theta') = \theta'$, and we have $^{-1}\theta'(\theta) = 0 \in H^{q+1}(\pi, n; G)$.

Proof. The fact that there exists a ${}^{-1}\theta' \in H^{q+1}(\pi', n'+1; G)$ such that ${}^{1}({}^{-1}\theta') = \theta'$ follows immediately from the suspension theorem of Eilenberg-MacLane [3] because $q < n + n' \leqq 2n' - 1$.

Let τ denote the transgression in the fibre space $\mathfrak{P} \to K(\pi, n)$. Since $i^*(\phi) = \theta'$, we have that $\tau(\theta') = 0$ by the exact sequence in [2]. However, since $q < n + n'$, we have $\tau(\theta'(\iota')) = {}^{-1}\theta'(\tau(\iota'))$, where $\iota' \in H^{n'}(\pi', n'; \pi')$ is the basic class. Hence $0 = \tau(\theta'(\iota')) = {}^{-1}\theta'(\tau(\iota')) = {}^{-1}\theta'(\theta)$.

We now wish to study the algebraic nature of the sets $\Phi(h) = \{\Phi_v(h)\}$, where Φ is a secondary cohomology operation. We will consider a homotopy associative principal fibre space (E, p, B, F, μ, h) with B $(n-1)$-connected and F $(n'-1)$-connected $(n' > n)$. If $q < n + n'$, it follows from the Künneth theorem that

$$H^q(F \times E; G) \approx H^q(F; G) \otimes H^0(E) + H^0(F) \otimes H^q(E; G),$$

and we will identify these groups.

LEMMA 3.3. *If $\phi \in H^q(E; G)$ with $q < n + n'$, then*

$$\mu^*(\phi) = i^*(\phi) \otimes 1 + 1 \otimes \phi,$$

where i: $F \to E$ is the inclusion.

Proof. From the defintion of a principal fibre space, we have

$$m^* i^*(\phi) = (1 \times i)^* \mu^*(\phi),$$

and the standard argument shows

$$m^* i^*(\phi) = i^*(\phi) \otimes 1 + 1 \otimes i^*(\phi)$$

since $q < n + n' < 2n'$.

We can write

$$\mu^*(\phi) = \alpha \otimes 1 + 1 \otimes \beta,$$

and we then see that $\alpha = i^*(\phi)$ and $i^*(\beta) = i^*(\phi)$.

On the other hand, since our fibre space is homotopy associative, we have

$$(j \times 1)^* \mu^*(\phi) = \pi^*(\phi) = 1 \otimes \phi. \quad \text{(See end of § 1.)}$$

But

$$(j \times 1)^*(i^*(\phi) \otimes 1 + 1 \otimes \beta) = 1 \otimes \beta$$

and thus $\beta = \phi$.

THEOREM 3.4. *Let $\mathfrak{P} = \mathfrak{P}(\pi, n; \pi', n', \theta)$ and let $\phi \in H^q(\mathfrak{P}; G)$ with $q < n + n'$. Let Φ be the secondary cohomology operation associated with ϕ. Let X be any space and let $h \in H^n(X; \pi)$ be such that $\theta(h) = 0 \in H^{n'+1}(X; \pi')$.*

Then

$$\Phi(h) \in H^q(X;G)/\theta' H^{n'}(X;\pi')$$

where $\theta' = i^*(\phi) \in H^q(\pi', n'; G)$.

Proof. If $v, v' \in \pi(X;\mathfrak{P})$ are such that $p_\#(v) = p_\#(v') = h$, then according to Lemma 1.2, there exists a $w \in H^{n'}(X;\pi')$ such that $\mu_\#(w,v) = v'$. It then follows from Lemma 1.3 and Lemma 3.3 that

$$\begin{aligned} v'^*(\phi) &= \mu_\#(w,v)^*(\phi) = (w,v)^*(\mu^*(\phi)) \\ &= (w,v)^*(i^*(\phi)\otimes 1 + 1\otimes\phi) \\ &= (w,v)^*(\theta'\otimes 1 + 1\otimes\phi) \\ &= w^*(\theta')\cup 1 + 1\cup v^*(\phi) \\ &= \theta'(w) + v^*(\phi). \end{aligned}$$

Thus

$$v'^*(\phi) - v^*(\phi) = \theta'(w) \in \theta' H^{n'}(X;\pi'),$$

which shows that any two element of $\Phi(h)$ differ by an element of $\theta' H^{n'}(X;\pi')$, and since $q < n + n' < 2n'$, θ' is additive and hence $\theta' H^{n'}(X;\pi')$ is a subgroup of $H^q(X;G)$. Furthermore, any element $w \in H^{n'}(X;\pi')$ gives rise to an element $v' \in \pi(X;\mathfrak{P})$ such that

$$p_\#(v') = p_\#(v) = h \text{ and } v'(\phi) = v^*(\phi) + \theta'(w).$$

Suppose now that $\mathfrak{P} = \mathfrak{P}(\pi, n; \pi', n', \theta)$ and $\phi \in H^{n+n'}(\mathfrak{P};G)$. It follows from the Künneth theorem that

$$\begin{aligned} H^{n+n'}(F\times\mathfrak{P};G) \approx{}& H^{n+n'}(F;G)\otimes H^0(\mathfrak{P}) \\ &+ H^0(F)\otimes H^{n+n'}(\mathfrak{P};G) + H^{n'}(F;H^n(\mathfrak{P};G)), \end{aligned}$$

where we have written F for $K(\pi', n')$. Now

$$H^{n'}(F;H^n(\mathfrak{P};G)) \approx \mathrm{Hom}(\pi';\mathrm{Hom}(\pi;G)) \approx \mathrm{Hom}(\pi'\otimes\pi;G).$$

Let $\mathfrak{P}' = \mathfrak{P}(\pi, n; \pi', n', \theta; G, n+n'-1, \phi)$. There is a Whitehead product homomorphism $\pi_{n'}(\mathfrak{P}')\otimes\pi_n(\mathfrak{P}') \to \pi_{n+n'-1}(\mathfrak{P}')$ which is an element $W \in \mathrm{Hom}(\pi'\otimes\&;G)$.

LEMMA 3.5.

$$\mu^*(\phi) = i^*(\phi)\otimes 1 + 1\otimes\phi + W.$$

Proof. For any space X and group G, we have a natural epimorphism $H^n(X;G) \to \mathrm{Hom}(H_n(X);G)$ carrying an element h into $h\vdash$. J. P. Meyer [18] has proved the following result: Let $\alpha \in \pi_{n'}(\mathfrak{P}')$, $\beta \in \pi_n(\mathfrak{P}')$, correspond to

$$\bar{\alpha} \in H_{n'}(\pi', n'; Z) \approx \pi_{n'}(K(\pi', n')) \approx \pi_{n'}(\mathfrak{P}) \approx \pi_{n'}(\mathfrak{P}')$$

and

$$\bar{\beta} \in H_n(\mathfrak{P}; Z) \approx H_n(\pi, n; Z) \approx \pi_n(K(\pi, n)) \approx \pi_n(\mathfrak{P}) \approx \pi_n(\mathfrak{P}')$$

respectively. We have the generalized Pontrjagin product $\bar{\alpha} * \bar{\beta} = \mu_*(\bar{\alpha} \times \bar{\beta}) \in H_{n+n'}(\mathfrak{P})$, where $\mu\colon K(\pi', n') \times \mathfrak{P} \to \mathfrak{P}$ is the multiplication. Then

$$\phi \vdash (\bar{\alpha} * \bar{\beta}) = [\alpha, \beta]$$

the Whitehead product of α and β.

The proof of Lemma 3.3 shows that the components of $\mu^*(\phi)$ in the first two terms of the direct sum decomposition are those indicated and we have only to identify the term in $H^{n'}(F; H^n(\mathfrak{P}; G))$. We have the diagram

$$\begin{tikzcd}
H_{n'}(F) \otimes H_n(\mathfrak{P}) \arrow[r, "P"] \arrow[dr] & H_{n+n'}(\mathfrak{P}) \arrow[r, "\phi\vdash"] & G \\
& H_{n+n'}(F \times \mathfrak{P}) \arrow[u, "\mu_*"] \arrow[ur, "\mu^*(\phi)\vdash"'] &
\end{tikzcd}$$

in which the unlabelled arrow denotes the cross-product, P denotes the Pontrjagin product, the left triangle is commutative by definition of P and the right one is commutative by naturality.

Now identifying $H_{n'}(F) \otimes H_n(\mathfrak{P})$ with $\pi' \otimes \pi$, the result of Meyer quoted above shows that going across the top line of our diagram is just W. It follows that

$$\begin{aligned} W(\bar{\alpha} \otimes \bar{\beta}) = \mu^*(\phi) \vdash (\bar{\alpha} \times \bar{\beta}) &= (i^*(\phi) \otimes 1 + 1 \otimes \phi + \chi) \vdash (\bar{\alpha} \times \bar{\beta}) \\ &= \chi(\bar{\alpha} \times \bar{\beta}), \end{aligned}$$

where χ is the component of $\mu^*(\phi)$ in $H^{n'}(F; H^n(\mathfrak{P}; G))$. It follows that $\chi = W$. (We have identified $H_{n'}(F) \otimes H_n(\mathfrak{P})$ as a direct summand of $H_{n+n'}(F \times \mathfrak{P})$ under the croos-product map.)

THEOREM 3.6.[2] *Let $\mathfrak{P} = \mathfrak{P}(\pi, n; \pi', n', \theta)$ and let $\phi \in H^{n+n'}(\mathfrak{P}; G)$. Let X be any space and let $h \in H^n(X; \pi)$ be such that $\theta(h) = 0 \in H^{n'+1}(X; \pi')$. Then*

$$\Phi(h) \in H^{n+n'}(X; G)/[\theta' + h \cup] H^{n'}(X; \pi'),$$

where $\theta' = i^*(\phi)$ and where the cup-product is relative to the Whitehead product pairing W in $\mathfrak{P}' = \mathfrak{P}(\pi, n; \pi', n', \theta; G, n + n' - 1, \phi)$.

---

[2] This theorem has also been obtained by M. G. Barratt and by J. P. Meyer.

Proof. As before, we consider $v, v' \in \pi(X;\mathfrak{P})$ such that $p_\#(v) = p_\#(v') = h$, and $w \in H^{n'}(X;\pi')$ such that $\mu_\#(w,v) = v'$. We then find

$$v'^*(\phi) = (w,v)^*(\theta' \otimes 1 + 1 \otimes \phi + W)$$

There is a commutative diagram

$$\begin{CD}
\mathrm{Hom}(\pi',\pi') \otimes \mathrm{Hom}(\pi,\pi) @>W_\#>> \mathrm{Hom}(\pi' \otimes \pi; G) \\
@V{\approx}VV @. \\
H^{n'}(F;\pi') \otimes H^n(\mathfrak{P};\pi) @. @VVV \\
@VVV @. \\
H^{n'}(F \times \mathfrak{P};\pi') \otimes H^n(F \times \mathfrak{P};\pi) @>>> H^{n+n'}(F \times \mathfrak{P}; G),
\end{CD}$$

where the bottom line is the cup product relative to W. Then if $\iota \in H^n(\mathfrak{P};\pi)$ and $\iota' \in H^{n'}(F;\pi')$ are the basic classes which we identify with their images in the cohomology of $F \times \mathfrak{P}$, and if $1' \in \mathrm{Hom}(\pi',\pi')$ and $1 \in \mathrm{Hom}(\pi,\pi)$ are the identity homomorphisms, then $1' \otimes 1$ corresponds to $\iota' \otimes \iota$. But $W_\#(1' \otimes 1) = W$ and hence W corresponds to $\iota' \cup \iota$. Thus

$$(w,v)^*(W) = (w,v)^*(\iota' \cup \iota) = w \cup v^*(\iota) = w \cup h.$$

Now as before,

$$v'^*(\phi) - v^*(\phi) = \theta'(w) + w \cup h \in [\theta' + h \cup]H^{n'}(X;\pi')$$

which is a subgroup since again $n + n' < 2n'$, and so θ' is additive. Furthermore, any element $w \in H^{n'}(X;\pi')$ gives rise to an element $v' \in \pi(X;\mathfrak{P})$ such that $p_\#(v') = p_\#(v) = h$ and $v'^*(\phi) = v^*(\phi) + \theta'(w) + w \cup h$.

4. Cohomology operations and obstructions. For simplicity, we treat only the case of secondary operations. The general case is similiar but more complicated. Let $\mathfrak{P} = \mathfrak{P}(\pi, n; \pi', n', \theta)$, let $\phi \in H^{n''+1}(\mathfrak{P};\pi'')$ and let $\mathfrak{P}' = \mathfrak{P}(\pi, n; \pi', n', \theta; \pi'', n'', \phi)$. Let X be any complex, X^i its i-skeleton, and consider a map $f\colon X^n \to \mathfrak{P}$ which can be extended to X^{n+1}, and hence to $X^{n'}$. Now let $\iota \in H^n(\mathfrak{P};\pi)$ be the basic class. The obstruction to extending f to a map $X^{n'+1} \to \mathfrak{P}$ is just $\theta(f^*(\iota))$. If we assume that $\theta(f^*\iota) = 0$, we can find extensions $f'\colon X^{n'+1} \to \mathfrak{P}$ of f and hence extensions $f'\colon X^{n''} \to \mathfrak{P}$ of f. For each such extension, there is defined an obstruction cocycle in $Z^{n''+1}(X;\pi'')$ and hence an obstruction class in $H^{n''+1}(X;\pi'')$. This class is not uniquely determined by f, but depends on the choice of an intermediate extension.

It is one of the classes $\Phi_v(f^*\iota)$ in § 3. The set $\Phi(f^*\iota)$ is the collection of all such obstruction classes for all possible intermediate extensions of f.

Chapter II. The Main Theorems.

5. Functional cohomology operations. In this section, we recall two definitions of functional primary cohomology operations and prove a theorem showing that they are the same.

Let $f\colon L \to K$ and let $\theta \in H^{n'+1}(\pi, n; \pi')$. Assuming f is an inclusion, we consider the following commutative diagram, where each row is the exact sequence of the pair (K, L):

$$\begin{tikzcd}
 & H^{n-1}(L;\pi) \arrow[r, "\delta"] \arrow[d, "{}^1\theta"] & H^n(K,L;\pi) \arrow[r, "j^*"] \arrow[d, "\theta"] & H^n(K;\pi) \arrow[r, "f^*"] \arrow[d, "\theta"] & H^n(L;\pi) \\
H^{n'}(K;\pi') \arrow[r, "f^*"] & H^{n'}(L;\pi') \arrow[r, "\delta"] & H^{n'+1}(K,L;\pi') \arrow[r, "j^*"] & H^{n'+1}(K;\pi'). &
\end{tikzcd}$$

Let $h \in H^n(K;\pi)$ be such that $\theta(h) = 0$ and $f^*(h) = 0$. By exactness, there exists an element $v \in H^n(K, L;\pi)$ such that $j^*(v) = h$. Since $j^*\theta(v) = \theta j^*(v) = \theta(h) = 0$, there exists an element $x \in H^{n'}(L;\pi')$ such that $\delta(x) = \pi(v)$. The set of all x such that $\delta(x) = \theta(v)$ with $j^*(v) = h$ is defined to be $\theta_f(h)$. in case θ is an additive cohomology operation, it is easy to see that $\theta_f(h)$ is a coset of $f^*(H^{n'}(K;\pi') + {}^1\theta(H^{n-1}(L;\pi)) \subset H^{n'}(L;\pi')$, and hence can be considered as an element in $H^{n'}(L;\pi')/\operatorname{Im} f^* + \operatorname{Im} {}^1\theta$.[3]

A second definition of functional primary cohomology operations is obtained by considering the following commutative diagram. Here the rows are exact sequences 3.1 applied to the fibration of the Postnikov space $\mathfrak{P} = \mathfrak{P}(\pi, n; \pi', n', \theta)$.

$$\begin{tikzcd}
 & H^{n'}(K;\pi') \arrow[r, "i_\#"] \arrow[d, "f^*"] & \pi(K;\mathfrak{P}) \arrow[r, "p_\#"] \arrow[d, "f^\#"] & H^n(K;\pi) \arrow[r, "\theta"] \arrow[d, "f^*"] & H^{n'+1}(K;\pi') \\
H^{n-1}(L;\pi) \arrow[r, "{}^1\theta"] & H^{n'}(L;\pi') \arrow[r, "i_\#"] & \pi(L;\mathfrak{P}) \arrow[r, "p_\#"] & H(L;\pi). &
\end{tikzcd}$$

Let $h \in H^n(K;\pi)$ be such that $f^*(h) = 0$ and $\theta(h) = 0$. As before, define $\bar\theta_f(h)$ to be the set of elements $x \in H^{n'}(L;\pi')$ such that $i_\#(x) = f^\#(v)$ with $p_\#(v) = h$. Again $\bar\theta_f(h)$ can be considered an element of

$$H^{n'}(L;\pi')/\operatorname{Im} {}^1\theta + \operatorname{Im} f^*.$$

---

[3] It is shown in [9] that this is true even when θ is not additive.

In [8] it is shown that there is an automorphism λ of a subgroup of $H^{n'}(L;\pi')/\operatorname{Im}{}^1\theta+\operatorname{Im}f^*$ such that $\lambda\bar{\theta}_f(h)=\theta_f(h)$ for all $u\in\operatorname{Ker}f^*\cap\operatorname{Ker}\theta$. We now strengthen this result.

THEOREM 5.1. $\bar{\theta}_f(h)=\theta_f(h)$.

Proof. This proof and most of the proofs to follow will use the method of universal examples. In this proof, we will show how the general result follows from the result in a universal example; later, these details will be omitted.

Special case. Let $L=K(\pi',n')$, $K=\mathfrak{P}$, $f=i\colon K(\pi',n')\to\mathfrak{P}$ be the inclusion of the fibre in the total space of the fibre space $p\colon\mathfrak{P}\to K(\pi,n)$, and let $h=[p]\in H^n(\mathfrak{P};\pi)$. Clearly $i^*([p])=0$ and $\theta([p])=0$. In the second definition, we may take $v=[1]$ because $p_\#([1])=[p]$. Also, $i^\#([1])=[i]=i_\#(\iota')$, where $\iota'\in H^{n'}(\pi',n';\pi')\approx\pi(K(\pi',n');K(\pi',n'))$ denotes the cohomology class corresponding to the identity map. Hence, $\bar{\theta}_i([p])=\{\iota'\}\in H^{n'}(\pi',n';\pi')/\operatorname{Im}i^*+\operatorname{Im}{}^1\theta$. It is well-known that if $r<n+n'$, then $p_1{}^*\colon H^r(\pi,n;G)\to H^r(\mathfrak{P},K(\pi',n');G)$ is an isomorphism and the following diagram is commutative:

$$\begin{tikzcd}
H^{r-1}(K(\pi',n');G) \arrow[r, "\tau"] \arrow[dr, "\delta"'] & H^r(\pi,n;G) \arrow[r, "p^*"] \arrow[d, "p_1{}^*"] & H^r(\mathfrak{P};G) \\
& H^r(\mathfrak{P},K(\pi',n');G), \arrow[ur, "j^*"'] &
\end{tikzcd}$$

where τ is the transgression homomorphism [2]. Since $n'+1<n+n'$ and $n<n+n'$, the diagram defining $\theta_i([p])$ may be replaced by the following diagram.

$$\begin{tikzcd}
& H^{n-1}(\pi',n';\pi) \arrow[r, "\tau"] \arrow[d, "{}^1\theta"] & H^n(\pi,n;\pi) \arrow[r, "p^*"] \arrow[d, "\theta"] & H^n(\mathfrak{P};\pi) \arrow[r, "i^*"] \arrow[d, "\theta"] & H^n(\pi',n';\pi) \\
H^{n'}(\mathfrak{P};\pi') \arrow[r, "i^*"] & H^{n'}(\pi',n';\pi') \arrow[r, "\tau"] & H^{n'+1}(\pi,n';\pi') \arrow[r, "p^*"] & H^{n'+1}(\mathfrak{P};\pi'). &
\end{tikzcd}$$

As above, let $\iota\in H^n(\pi,n;\pi)$ be the fundamental class. Then $p^*(\iota)=[p]$, and by definition of $\mathfrak{P}$, $\theta(\iota)=\tau(\iota')$. Hence

$$\theta_i([p])=\{\iota'\}=\bar{\theta}_i([p])\in H^{n'}(\pi',n;\pi')/\operatorname{Im}i^*+\operatorname{Im}{}^1\theta.$$

General Case. Let $f\colon L\to K$, $h\in H^n(K;\pi)$, $\theta(h)=0$, and $f^*(h)=0$. Let $\chi(h)\colon K\to K(\pi,n)$ represent h. Since $\theta(h)=0$, there exists a map

$\bar{\chi}(h): K \to \mathfrak{P}$ such that $p_\#[\bar{\chi}(h)] = [\chi(h)] = h$. Since $f^*(h) = 0$, there exists a map $\tilde{\chi}(h): L \to K(\pi', n')$ such that the following diagram is commutative up to homotopy:

$$\begin{tikzcd} L \arrow[r, "\tilde{\chi}(h)"] \arrow[d, "f"'] & K(\pi', n') \arrow[d, "i"] \\ K \arrow[r, "\bar{\chi}(h)"] & \mathfrak{P}. \end{tikzcd}$$

By an obvious naturality condition for functional cohomology operations, we have

$$\begin{aligned} \theta_f(h) &= \theta_f(\chi(h)^*([p])) = \tilde{\chi}(h)^*\theta_i([p]) \\ &= \tilde{\chi}(h)^*\bar{\theta}_i([p]) = \bar{\theta}_f(\bar{\chi}(h)^*([p])) = \bar{\theta}_f(h). \end{aligned}$$

6. The two formulas. In this section, we state and prove our two formulas relating secondary cohomology operations and primary cohomology operations. To avoid technical complications, we restrict ourselves in this section to stable secondary cohomology operations of a single variable. More general situations will be discussed in Sections 7 and 8.

As in Section 3, let $\theta \in H^{n'+1}(\pi, n; \pi')$, $\mathfrak{P} = \mathfrak{P}(\pi, n; \pi', n'; \theta)$, and let $\phi \in H^q(\mathfrak{P}; G)$ define the secondary cohomology operation Φ. Let $\theta'(\iota') = i^*(\phi) \in H^q(\pi', n'; G)$. We will assume that $q < n + n'$ throughout this section.

THEOREM 6.1. *Let $f: L \to K$ and $h \in H^n(K; \pi)$ be such that $f^*(h) = 0$ and $\theta(h) = 0$. Then*[4]

$$f^*\Phi(h) = \theta'(\theta_f(h)) \in H^q(L; G)/\theta' f^*(H^{n'}(K; \pi')).$$

Proof. We give a proof in the universal example. The general case then follows as in the proof of Theorem 5.1. Let $L = K(\pi', n')$, $K = \mathfrak{P}$, $h = [p]$, and $f = i$. Since $p_\#[1] = [p]$, $\Phi([p]) = \{\phi_\#([1])\} = \{\phi\}$. However, as in the proof of Theorem 5.1, $\theta_i([p]) = \{\iota'\}$, and hence $\theta'(\theta_i([p]) = \theta'\{\iota'\} = i^*\Phi([p])$. To complete the proof, we must show that both sides are well-defined in general. According to Theorem 3.4, under the assumption $q < n + n'$, $\Phi(h) \in H^q(K; G)/\theta'(H^{n'}(K; \pi'))$. Hence

$$f^*\Phi(h) \in H^q(L; G)/\theta' f^*(H^{n'}(K; \pi')).$$

[4] Special cases of this formula were known to Shimada [12] and Stein [13].

The right hand side is defined modulo

$$\theta'(f^*(H^{n'}(K;\pi'))+{}^1\theta(H^{n-1}(L;\pi))=\theta' f^*(H^{n'}(K;\pi'))$$

because $\theta'{}^1\theta=0$ by Lemma 3.2.

Remark. Since Φ is natural, under the hypothesis of Theorem 6.1, we have $f^*\Phi(h)=\Phi(f^*(h))=\Phi(0)=0$. Here $0\in H^q(L;G)/\theta'(H^{n'}(L;\pi'))$. Our theorem gives more delicate information as we factor out by the smaller subgroup $\theta' f^*(H^{n'}(K;\pi'))$ and show that $f^*\Phi(h)$ is the image under θ' of a particular element, $\theta_f(h)$.

Before stating our second formula, we prove the following lemma.

LEMMA 6.2. *Let $p:E\to B$ be a fibre space with fibre F and let $i:F\to E$ be the inclusion. Let $h\in H^{n'+1}(B;\pi')$ be such that $\tau(v)=h$, where $v\in H^{n'}(F;\pi')$ and τ is the transgression. Let $\psi\in H^{q+1}(\pi',n'+1;G)$ be such that $\psi(h)=0$. Then*

$$i^*((\psi_p)(h))={}^1\psi(v)\in H^q(F;G)/{}^1\psi i^*(H^{n'}(E;\pi')).$$

Proof. Let $\chi(h):B\to K(\pi',n'+1)$ represent h. Since $\psi(h)=0$, there exist a map $\bar\chi(h):B\to\mathfrak{P}(\pi',n'+1;G,q,\psi)=\mathfrak{P}'$ such that $p'_\#(\bar\chi(h))=[\chi(h)]=h$, where $p':\mathfrak{P}'\to K(\pi',n'+1)$. Since $p^*(h)=0$, there exists a map $\tilde\chi(h):E\to K(G,q)$ such that the following diagram is commutative up to homotopy:

$$\begin{tikzcd} E \arrow[r, "\tilde\chi(h)"] \arrow[d, "p"] & K(G,q) \arrow[d, "i'"] \\ B \arrow[r, "\bar\chi(h)"] & \mathfrak{P}' \end{tikzcd}$$

We may assume that $i':K(G,q)\to\mathfrak{P}'$ is a fibre map and that $\bar\chi(h)$ is a fibre preserving map. Hence $\tilde\chi(h)\,|\,F:F\to K(\pi',n')$ and $(\tilde\chi(h)\,|\,F)^*(\iota')=\bar v$ is such that $\tau(\bar v)=\tau(\tilde\chi(h)\,|\,F)^*(\iota')=\bar\chi(h)^*(\tau(\iota'))=\bar\chi(h)^*(\iota'_{n'+1})=h$. Moreover, because $\bar v-v=i^*(x)$ for some $x\in H^{n'}(E;\pi')$,

$${}^1\psi(\bar v)={}^1\psi(v)\in H^q(F;G)/\mathrm{Im}\,{}^1\psi i^*(H^{n'}(E;\pi')),$$

we need only prove our lemma for $\bar v$. Hence by naturality, it is sufficient to prove the lemma for the fibre space $i':K(G,q)\to\mathfrak{P}'$. As in the proof of Theorem 5.1, $(\psi)_{i'}([p])=\iota''$, and since $i'':K(\pi',n')\to K(G,q)$, the inclusion of the fibre in the fibre space $i':K(G,q)\to\mathfrak{P}'$, is the loop functor applied to $\psi:K(\pi',n'+1)\to K(G,q+1)$, we have that $(i'')^*(\iota'')={}^1\psi(\iota')$ and our lemma is proved.

We now prove our second formula. As above, let $\theta \in H^{n'+1}(\pi, n; \pi')$, $\mathfrak{P} = \mathfrak{P}(\pi, n; \pi', n', \theta)$, and let $\phi \in H^q(\mathfrak{P}; G)$ define the secondary operation Φ. Let $\theta'(\iota') = i^*(\phi) \in H^q(\pi', n'; G)$. Because $q < n + n'$, $\theta' = {}^1\psi$, where $\psi \in H^{q+1}(\pi', n' + 1; G)$.

THEOREM 6.3. *Let $f\colon L \to K$ and $h \in H^n(K; \pi)$ be such that $f^*\theta(h) = 0$. Then*

$$\Phi(f^*(h)) = \psi_f(\theta(h)) \in H^q(L; G)/f^*(H^q(K; G)) + \theta'(H^{n'}(L; \pi')).$$

Proof. We give this proof in the universal example. Let $L = \mathfrak{P}$, $K = K(\pi, n)$, $f = p$, and $h = \iota$. Then $\Phi(p^*(\iota)) = \phi$. By Lemma 6.2, with $h = \theta(\iota)$ we have $i^*((\psi)_p(\theta(\iota))) = \theta'(\iota') = i^*(\phi)$. However, under the dimensional restriction $q < n + n'$, an element in $H^q(\mathfrak{P}; G)$ is determined, modulo $p^*(H^q(\pi, n; G))$, by its image under i^*. Hence $(\psi)_p(\theta(\iota)) = \phi = \Phi(p^*(\iota)) \in H^q(\mathfrak{P}; G)/p^*(H^q(\pi, n; G)) + \theta'(H^{n'}(\mathfrak{P}; \pi'))$. To pass to the general case, it only has to be noted that both sides are well-defined.

Remark. The formulas in Theorems 6.1 and 6.3 are dual to each other. Notice that in Theorem 6.1, $h \in \operatorname{Ker} f^* \cap \operatorname{Ker} \theta$ and the values are in Coker $f^*\theta'$, while in Theorem 6.3, $h \in \operatorname{Ker} f^*\theta$ and the values are in Coker $(f^* + \theta')$. We hope to make this duality more precise and give some applications in a later paper.

7. A non-stable case. In this section, we generalize the formulas of the preceding section to some secondary cohomology operations where $q \geqq n + n'$. An example of this section will be discussed in detail in Section 10.

The condition $q < n + n'$ was used in the proof of Theorem 6.1 only to show that both sides of the formula were well-defined. Let $L(\phi, h)$ be the subgroup generated by all differences $\Phi_v(h) - \Phi_{v'}(h)$ for $p_\#(v) = p_\#(v') = h$. Then $\Phi(h)$ can be considered as an element of $H^q(K; G)/L(\phi, h)$ for $h \in H^n(K; \pi)$ with $\theta(h) = 0$. The proof of the following theorem is the same as that of Theorem 6.1.

THEOREM 7.1. *Let $f\colon L \to K$ and $h \in H^n(K; \pi)$ be such that $f^*(h) = 0$ and $\theta(h) = 0$. Assume that θ' is additive. Then*

$$f^*\Phi(h) = \theta'(\theta_f(h)) \in H^q(L; G)/f^*(L(\phi, h) + \theta' f^*(H^{n'}(K; \theta'))$$
$$+ \theta'^1\theta(H^{n-1}(L; \pi)).$$

We now discuss the case $q = n + n'$ for Theorem 6.3. Let $\theta \in H^{n'+1}(\pi, n; \pi')$. Assume that there exists an element $\phi \in H^q(\mathfrak{P}; G)$ such that $\theta' = i^*(\phi) = {}^1\psi$,

where $\psi \in H^{q+1}(\pi', n'+1; G)$ is such that $\psi(\theta) = \iota \cup \theta \in H^{q+1}(\pi, n; G)$ and the cup product is with respect to the Whitehead product pairing $\pi \otimes \pi' \to G$ (see Theorem 3.6).

THEOREM 7.2. *Let $f\colon L \to K$ and $h \in H^n(K;\pi)$ be such that $f^*\theta(h) = 0$. Then*

$$\Phi(f^*(h)) = (\psi - h \cup)_f(\theta(h)) \in H^q(L;G)/f^*(H^q(K;G)) + (\theta' - f^*(h) \cup)(H^{n'}(L;\pi')).$$

Before giving the proof of this theorem, we must discuss the definition of $(\psi - h \cup)_f$; it will be analogous to the second definition in Section 5.

Let $f\colon L \to K$, $h \in H^n(K;\pi)$, $v \in H^{n'+1}(K;\pi')$, and $\psi \in H^{q+1}(\pi', n'+1; G)$ be such that $f^*(v) = 0$ and $\psi(v) = h \cup v$. To define

$$(\psi - h \cup)_f(v) \in H^q(L;G)/f^*(H^q(K;G)) + ({}^1\psi - f^*(h)\cup)(H^{n'}(L;\pi')),$$

we study the universal example. Let

$$K = \mathfrak{P}(\pi, n; \pi', n'+1, 0; G, q, \psi(\iota') - \iota \cup \iota') = \bar{\mathfrak{P}},$$

let $L = \mathfrak{P}(\pi, n; \pi', n'+1, 0; G, q, \psi(\iota') - \iota \cup \iota'; \pi', n', \iota') = \mathfrak{P}'$, and let $p\colon \mathfrak{P}' \to \bar{\mathfrak{P}}$ be the fibre map. Consider the spectral sequence of this fibring. The element $\iota \otimes \iota' \in E_2^{n,n'}$ kills the element $\iota \cup \iota' = \psi(\iota') \in E_2^{n+n'+1,0}$ because $d^{n'+1}(\iota \otimes {}^1\iota') = \iota \cup \iota'$. Hence all (d^r)'s are 0 on the element ${}^1\psi({}^1\iota') \in E_2^{0,n+n'}$. This element gives rise to an element in $H^{n+n'}(\mathfrak{P}';G)$, defined modulo $p^*H^q(\bar{\mathfrak{P}};G)$, which we define to be $(\psi - \iota \cup)_p(\iota')$. The general definition is obtained by mapping into the universal example.

The proof of Lemma 6.2 now generalizes to a proof of the following lemma:

LEMMA 7.3. *Let $p\colon E \to B$ be a fibre space with fibre F and let $i\colon F \to E$ be the inclusion. Let $h \in H^n(B;\pi)$, $v \in H^{n'+1}(B;\pi')$, and $\psi \in H^{q+1}(\pi', n'+1; G)$ be such that $\psi(v) = h \cup v$ and $\tau(w) = v$, where $w \in H^{n'}(F;\pi')$. Then*

$$i^*(\psi - h \cup)_p(v) = {}^1\psi(w) \in H^q(F;G)/{}^1\psi i^*(H^{n'}(E;\pi')).$$

With this lemma, the proof of Theorem 6.3 easily generalizes to a proof of Theorem 7.2.

8. Cohomology operations on several variables. Our two formulas have analogs for secondary cohomology operations on more than one variable. In this section, we give the analog for the Massey triple product [15].

Let $u \in H^p(K)$, $v \in H^q(K)$, and $w \in H^r(K)$, where the coefficients are

in a commutative ring with unit. If $u \cup v = 0$ and $v \cup w = 0$, then the triple product $\langle u, v, w\rangle \in H^{p+q+r-1}(K)/u \cup H^{q+r-1}(K) + H^{p+q-1}(K) \cup v$ is defined as follows. Let $\bar{u}$, $\bar{v}$, and $\bar{w}$ be cocycles representing u, v, and w respectively. Let $\delta a = \bar{u} \cup \bar{v}$, $\delta b = \bar{v} \cup \bar{w}$. Then a representative of $\langle u, v, w\rangle$ is $a \cup \bar{w} + (-1)^{p+1}\bar{u} \cup b$.

We now define left and right functional cup products. Let $f\colon L \to K$, $u \in H^p(K)$, and $v \in H^q(K)$. If $u \cup v = 0$ and $f^*(u) = 0$, then

$$u \overset{L}{\underset{f}{\cup}} v \in H^{p+q-1}(L)/f^*H^{p+q-1}(K) + H^{p-1}(L) \cup f^*(v)$$

is defined as follows: Let $\delta c = f^{\#}(\bar{u})$, where $f^{\#}$ is the cochain map induced by f. Then a representative of $u \overset{L}{\underset{f}{\cup}} v$ is $f^{\#}(a) - c \cup f^{\#}(\bar{v})$. Also, if $u \cup v = 0$ and $f^*(v) = 0$, then $u \overset{R}{\underset{f}{\cup}} v \in H^{p+q-1}(L)/f^*(H^{p+q-1}(K)) + f^*(u) \cup H^{q-1}(L)$ is defined as follows: Let $\delta d = f^{\#}(\bar{v})$. Then a representative of $u \overset{R}{\underset{f}{\cup}} v$ is $f^{\#}(a) + (-1)^{p+1}f^{\#}(\bar{u}) \cup d$.

The analog of Theorem 6.1 is the following theorem.

THEOREM 8.1. *Let $u \in H^p(K)$, $v \in H^q(K)$, $w \in H^r(K)$, and $f\colon L \to K$ be such that $f^*(u) = 0$, $u \cup v = 0$, and $v \cup w = 0$. Then*[5]

$$f^*\langle u, v, w\rangle = (u \overset{L}{\underset{f}{\cup}} v) \cup f^*(w) \in H^{p+q+r-1}(L)/f^*(H^{q+r-1}(K) \cup w).$$

Proof. Let $\delta a = \bar{u} \cup \bar{v}$, $\delta b = \bar{v} \cup \bar{w}$, *and* $\delta c = f^{\#}(\bar{u})$. Then $f^*\langle u, v, w\rangle$ has as a representative

$$\begin{aligned} &f^{\#}(a) \cup f^{\#}(\bar{w}) + (-1)^{p+1}f^{\#}(\bar{u}) \cup f^{\#}(b) \\ &\qquad = f^{\#}(a) \cup f^{\#}(\bar{w}) + (-1)^{p+1}\delta c \cup f^{\#}(b). \end{aligned}$$

On the other hand, $(u \overset{L}{\underset{f}{\cup}} v) \cup f^*(w)$ has a representative

$$f^{\#}(a) \cup f^{\#}(\bar{w}) - c \cup f^{\#}(\bar{v}) \cup f^{\#}(\bar{w}) = f^{\#}(a) \cup f^{\#}(\bar{w}) - c \cup \delta f^{\#}(b).$$

However, $\delta(c \cup f^{\#}(b)) = \delta c \cup f^{\#}(b) + (-1)^{p-1}c \cup \delta f^{\#}(b)$, and hence the representatives of $f^*\langle u, v, w\rangle$ and $(u \overset{L}{\underset{f}{\cup}} v) \cup f^*(w)$ differ by $\delta((-1)^{p+1}c \cup f^{\#}(b))$.

The analog of Theorem 6.3 is the following theorem.

THEOREM 8.2. *Let $u \in H^p(K)$, $v \in H^q(K)$, $w \in H^r(K)$, and $f\colon L \to K$ be such that $f^*(u \cup v) = 0$, $f^*(v \cup w) = 0$, and $u \cup v \cup w = 0$. Then*

[5] Formulas similar to this appear in [15].

$$\langle f^*(u), f^*(v), f^*(w)\rangle$$

$$= u \overset{R}{\underset{f}{\cup}} (v \cup w) - (u \cup v) \overset{L}{\underset{f}{\cup}} w \in H^{p+q+r-1}(L)/f^*(u) \cup H^{q+r-1}(L)$$

$$+ H^{p+q-1}(L) \cup f^*(w) + f^*(H^{p+q+r-1}(K)).$$

Proof. Let $\delta a = f^\#(\bar{u} \cup \bar{v}) = f^\#(\bar{u}) \cup f^\#(\bar{v})$, $\delta b = f^\#(\bar{v} \cup \bar{w})$, and $\delta c = \bar{u} \cup \bar{v} \cup \bar{w}$. Then $\langle f^*(u), f^*(v), f^*(w)\rangle$ has as a representative

$$(-1)^{p+1} f^\#(\bar{u}) \cup b + a \cup f^\#(\bar{w}).$$

However, $u \overset{R}{\underset{f}{\cup}} (v \cup w) - (u \cup v) \overset{L}{\underset{f}{\cup}} w$ has as a representative

$$f^\#(c) + (-1)^{p+1} f^\#(\bar{u}) \cup b - (f^\#(c) - a \cup f^\#(\bar{w}))$$

$$= (-1)^{p+1} f^\#(\bar{u}) \cup b + a \cup f^\#(\bar{w}).$$

The generalizations of these formulas to n-tuple products are straightforward and are left to the reader.

Chapter III. Applications.

9. The Adem operation. ([1]) Let $\mathfrak{P}_n = \mathfrak{P}(Z, n; Z_2, n+1, Sq^2\iota)$ with $n \geqq 4$. (Only a slight change is necessary to handle the case $n = 3$.) $\mathfrak{P}_n$ is a fibre space over $K(Z, n)$ with fibre $K(Z_2, n+1)$. We consider the mod 2 cohomology of $\mathfrak{P}_n$. Let $\iota \in H^n(Z, n; Z_2) \approx Z_2$ be the generator. Then $H^{n+1}(Z, n; Z_2) = 0$, $H^{n+2}(Z, n; Z_2) \approx Z_2$ generated by $Sq^2\iota$, $H^{n+3}(Z, n; Z_2) \approx Z_2$ generated by $Sq^3\iota$, and $H^{n+4}(Z, n; Z_2) \approx Z_2$ generated by $Sq^4\iota$. Similarly, if ι' generates $H^{n+1}(Z_2, n+1; Z_2) \approx Z_2$, then $H^{n+2}(Z_2, n+1; Z_2) \approx Z_2$ generated by $Sq^1\iota'$, $H^{n+3}(Z_2, n+1; Z_2) \approx Z_2$ generated by $Sq^2\iota'$. Furrthermore, we have $\tau(\iota') = Sq^2\iota$, and hence $\tau(Sq^1\iota') = Sq^1Sq^2\iota = Sq^3\iota$ and $\tau(Sq^2\iota') = Sq^2Sq^2\iota = Sq^3Sq^1\iota = 0$. It follows that $H^i(\mathfrak{P}_n; Z_2) = 0$ for $i < n$, $H^n(\mathfrak{P}_n; Z_2) \approx Z_2$ generated by $p^*\iota$, $H^{n+1}(\mathfrak{P}_n; Z_2) = 0$, $H^{n+2}(\mathfrak{P}_n; Z_2) = 0$, and $H^{n+3}(\mathfrak{P}_n; Z_2) \approx Z_2$ generated by an element ϕ_n such that $i^*(\phi_n) = Sq^2\iota' \in H^{n+3}(Z_2, n+1; Z_2)$. These results all follow from the exact sequence of Cartan-Serre [2]. ϕ_n gives rise to a secondary cohomology operation Φ_n which is defined, for any space X, on those elements $h \in H^n(X; Z)$ such that $Sq^2h = 0 \in H^{n+2}(X; Z)$, and since $n \geqq 3$, we can apply Theorem 3.4 to see that Φ_n takes its values in $H^{n+3}(X; Z_2)/Sq^2H^{n+1}(X; Z_2)$.

As the first application of our earlier results, we prove a formula which occurred in computations Peterson has made of homotopy groups of the unitary groups. The proof illustrates a useful way to apply our main

theorems to compute secondary cohomology operations. Namely, to compute a secondary operation Φ on a class h in a space X, construct a space Y and a map $f\colon Y \to X$ so that the induced cohomology map is a monomorphism in the dimension of $\Phi(h)$, and $f^*(h) = 0$. Such a space can often be constructed as a fibre space over X with a $K(\pi, n)$ as fibre and h as k-invariant. Then apply Theorem 6.1.

THEOREM 9.1. *Let $\iota \in H^4(Z, 4; Z) \approx Z$ be a generator and let δ^* be the Bockstein operator associated with the exact sequence $0 \to Z \to Z \to Z_2 \to 0$. Then*

$$\Phi_7(\delta^* Sq^2 \iota) = 0 \in H^{10}(Z, 4; Z_2).$$

Proof. We note first that $H^8(Z, 4; Z_2) \approx Z_2$ is generated by ι^2 and that $Sq^2 \iota^2 = 0$, so that $Sq^2 H^8(Z, 4; Z_2) = 0$, and hence Φ_7 does take its values in $H^{10}(Z, 4; Z_2)$ rather than in a quotient of this group.

Let $\mathfrak{P} = \mathfrak{P}(Z, 4; Z, 6, \delta^* Sq^2 \iota)$, and let $f\colon \mathfrak{P} \to K(Z, 4)$ be the fibre map. A straightforward calculation with the spectral sequence of f shows that $H^{10}(\mathfrak{P}; Z_2) \approx Z_2 + Z_2 + Z_2$, and the generators are $f^*(\iota Sq^2 \iota)$, $f^*(Sq^4 Sq^2 \iota)$, and an element α such that $i^*(\alpha) = Sq^4 \iota' \in H^{10}(Z, 6; Z_2)$, where $i\colon K(Z, 6) \to \mathfrak{P}$ is the inclusion of the fibre, and ι' generates $H^6(Z, 6; Z_2)$. Thus f^* is a monomorphism on $H^{10}(Z, 4; Z_2)$. We apply Theorem 6.1 and find that

$$f^* \Phi_7(\delta^* Sq^2 \iota) = Sq^2 Sq^2{}_f \delta^* Sq^2 \iota \in H^{10}(\mathfrak{P}; Z_2)/Sq^2 f^* H^8(Z, 4; Z_2).$$

But $Sq^2 f^* \iota^2 = 0$, so the relation holds in $H^{10}(\mathfrak{P}; Z_2)$. Furthermore, $i^*(Sq^2 Sq^2{}_f \delta^* Sq^2 \iota) = Sq^2 i^*(Sq^2{}_f \delta^* Sq^2 \iota) \in Sq^2 H^8(Z, 6; Z_2)$. But $H^8(Z, 6; Z_2)$ is generated by $Sq^2 \iota'$ and $Sq^2 Sq^2 \iota' = Sq^3 Sq^1 \iota' = 0$ so that $i^*(Sq^2 Sq^2{}_f \delta^* Sq^2 \iota) = 0$. It follows that

$$Sq^2 Sq^2{}_f \delta^* Sq^2 \iota = a(f^* Sq^4 Sq^2 \iota) + b(f^*(\iota Sq^2 \iota)),$$

where a and b are either 0 or 1.

Let $\mu = Sq^2{}_f \delta^* Sq^2 \iota \in H^8(\mathfrak{P}; Z_2)$. We have

$$Sq^2 \mu(f^* \iota) = a(Sq^4 Sq^2 f^* \iota) + b(f^* \iota)(Sq^2 f^* \iota).$$

Consider the space $S^2(CP(4)) = S^4 \cup e^6 \cup e^8 \cup e^{10}$, where $CP(4)$ is complex-projective 4-space. We have $Sq^2\{S^4\} = \{t^6\}$ and $Sq^4\{e^6\} = \{e^{10}\}$. Also $\{S^4\} \cup \{e^6\} = 0$ since cup-products are zero in a suspension. It follows from Lemma 9.2 which we will prove below that $\mu\{S^4\} = 0$. Thus

$$0 = Sq^2 \mu\{S^4\} = a\{e^{10}\} + 0,$$

so that $a = 0$.

Now let $Y = S^4 \cup e^6$, where e^6 is attached by $\eta_4 \neq 0 \in \pi_5(S^4)$. According to Theorem 1.2 of [5], there is a complex $X = Y \cup e^{10}$ such that $\{S^4\} \cup \{e^6\} = \{e^{10}\}$ if and only if $[\eta_4, \iota_4]$ is in the image of $(\eta_4)_*: \pi_8(S^5) \to \pi_8(S^4)$. On the other hand, we have, according to [14],

$$[\eta_4, \iota_4] = \alpha_4 \circ \eta_7 = \eta_4 \circ \nu_5 = (\eta_4)_*(\nu_5).$$

Hence there is a space X of the right type. We have $H^8(X; Z_2) = 0$, so that $\mu\{S^4\} = 0$. Thus $0 = Sq^2\mu\{S^4\} = 0 + b\{e^{10}\}$, so that $b = 0$.

This proves the theorem.

LEMMA 9.2. *The double suspension of μ, ${}^2\mu = 0$ as a cohomology operation.*

Proof. We consider the path space over $\mathfrak{P}$ with fibre ${}^1\mathfrak{P}$ and compute the mod 2 cohomology spectral sequence. It is easy to see that up to dimension 7 the only non-zero groups $H^i({}^1\mathfrak{P}; Z_2)$ are Z_2 in dimensions 3, 5, and 7 generated by classes whose transgressions are respectively ι, $Sq^2\iota$, and μ, i.e. by the classes ${}^1\iota$, ${}^1(Sq^2\iota)$, ${}^1\mu$. We next consider the path space over ${}^1\mathfrak{P}$ with fibre ${}^2\mathfrak{P}$ and compute in the same way. We note that ${}^2\mathfrak{P}$ has homotopy groups Z in dimensions 2 and 4 and all others trivial. Since $H^5(Z,2;Z) = 0$, we must have ${}^2\mathfrak{P} = K(Z,2) \times K(Z,4)$. Thus the mod 2 cohomology of ${}^2\mathfrak{P}$ in dimensions up to 6 has as an additive basis $\{\alpha, \beta, \alpha^2, \alpha^3, \alpha\beta, Sq^2\beta\}$, where $\dim \alpha = 2$ and $\dim \beta = 4$. Then we must have $\tau(\alpha) = {}^1\iota$, $d_3(\beta) = {}^1\iota \otimes \alpha$, $\tau(\alpha^2) = \tau(Sq^2\alpha) = Sq^2({}^1\iota)$, $d_3(\alpha^3) = {}^1\iota \otimes \alpha^2$, $d_3(\alpha\beta) = {}^1\iota \otimes \beta + \alpha \otimes {}^1\iota \otimes \alpha = {}^1\iota \otimes \beta + {}^1\iota \otimes \alpha^2$, and hence $\tau(Sq^2\beta) = {}^1\mu$. This means ${}^2\mu = Sq^2\beta$. But in the cohomology operation corresponding to ${}^2\mu$, we must factor out the image of Sq^2. Hence this operation is zero.

10. The operation ${}^n\Psi_n$. ([13])[6]

Let $\mathfrak{P}_n = \mathfrak{P}(Z,2;Z,2n+1,C^{n+1})$, where C^{n+1} is the $(n+1)$-st power of the basic class $C \in H^2(Z,2;Z)$. Let $p: \mathfrak{P}_n \to K(Z,2)$ be the fibre map, and let $\iota \in H^2(Z,2;Z_2)$ and $\iota' \in H^{2n+1}(Z,2n+1;Z_2)$ be generators. Then $d_{2n+1}(\iota') = \tau(\iota') = \iota^{n+1}$ so that $d_{2n+1}(\iota \otimes \iota') = \iota^{n+2}$, and

$$d_{2n+3}(Sq^2\iota') = \tau(Sq^2\iota') = Sq^2\tau(\iota') = Sq^2(\iota^{n+1}) = \begin{cases} 0, n\text{: odd} \\ \iota^{n+2}\, n\text{: even.} \end{cases}$$

Since ι^{n+2} is in the image of d_{2n+1}, we have in any case

$$\tau(Sq^2\iota') = 0 \in E_{2n+4}{}^{2n+4,0}.$$

[6] Much of the material of this section is taken from [13].

It follows that $H^{2n+3}(\mathfrak{P}_n; Z_2) \approx Z_2$ and a generator ψ_n of this group has the property that $i^*(\psi_n) = Sq^2\iota' \in H^{2n+3}(Z, 2n+1; Z_2)$. Let Ψ_n be the secondary cohomology operation associated with ψ_n. Then Ψ_n is defined for any space X on those elements $h \in H^2(X)$ such that $h^{n+1} = 0 \in H^{2n+2}(X)$, and according to Theorem 3.6,

$$\Psi_n(h) \in H^{2n+3}(X; Z_2)/[Sq^2 + h \cup] H^{2n+1}(X; Z),$$

where the coefficient pairing $Z \otimes Z \to Z_2$ which defines the cup product is the Whitehead product pairing $\pi_{2n+1}(\mathfrak{P}_n') \otimes \pi_2(\mathfrak{P}_n') \to \pi_{2n+2}(\mathfrak{P}_n')$ in

$$\mathfrak{P}_n' = \mathfrak{P}_n(Z, 2; Z, 2n+1, C^{n+1}; Z_2, 2n+2, \psi_n).$$

LEMMA 10.1. *Let $P_n(C)$ denote complex-projective space of n complex dimensions. Then ψ_n is the second Postnikov invariant $k^{2n+3}(P_n(C))$ of $P_n(C)$.*

Proof. Obvious.

It follows from Lemma 10.1 that the Whitehead product

$$\pi_{2n+1}(\mathfrak{P}_n') \otimes \pi_2(\mathfrak{P}_n') \to \pi_{2n+2}(\mathfrak{P}_n')$$

is the same as the Whitehead product

$$\pi_{2n+1}(P_n(C)) \otimes \pi_2(P_n(C)) \to \pi_{2n+2}(P_n(C)).$$

This product has been computed (see [13] or [17]) and is trivial for n odd and non-trivial for n even. Thus when n is odd, we see that the cup-product term in the denominator for Ψ_n vanishes, and we have

$$\Psi_n(h) \in H^{2n+3}(X; Z_2)/Sq^2 H^{2n+1}(X; Z), \qquad n: \text{odd},$$

while for n even, the cup-product is relative to the non-trivial coefficient pairing $Z \otimes Z \to Z_2$.

The next theorem illustrates one of the ways our main theorems can be used to find relations between primary and secondary cohomology operations, i.e., the computation of primary operations in the cohomology of spaces with two homotopy groups.

THEOREM 10.2. *Let X be a space, let n be odd, and let $h \in H^2(X)$ be such that $h^{n+1} = 0$. Then*

$$Sq^2\Psi_n(h) = 0 \in H^{2n+5}(X; Z_2).$$

Proof. Let $f: X \to K(Z, 2)$ be a map such that $f^*(\iota) = h$, where $\iota \in H^2(Z, 2; Z)$ is the basic class. Then according to Theorem 7.2,

$$
\begin{aligned}
\Psi_n(h) &= Sq_f^2(h^{n+1}) \\
&\in H^{2n+3}(X;Z_2)/(f^*H^{2n+3}(Z,2;Z_2)+Sq^2H^{2n+1}(X;Z)) \\
&= H^{2n+3}(X;Z_2)/Sq^2H^{2n+1}(X;Z)
\end{aligned}
$$

since the coefficient pairing in the cup-product term in Theorem 7.2 is trivial. Now

$$
\begin{aligned}
Sq^2\Psi_n(h) &\in H^{2n+5}(X;Z_2)/Sq^2Sq^2H^{2n+1}(X;Z) \\
&= H^{2n+5}(X;Z_2)/Sq^3Sq^1H^{2n+1}(X;Z) = H^{2n+5}(X;Z_2).
\end{aligned}
$$

Furthermore, according to Theorem 7.1, (Φ_{2n+2} is the Adem operation of § 9) $Sq^2\Psi_n(h) = Sq^2Sq^2{}_f(h^{n+1}) = f^*\Phi_{2n+2}(\iota^{n+1})$ in

$$
\begin{aligned}
&H^{2n+5}(X;Z_2)/(f^*Sq^2H^{2n+3}(Z,2;Z_2)+Sq^2f^*H^{2n+3}(Z,2;Z_2)+Sq^2Sq^2H^{2n+1}(X;Z)) \\
&\qquad = H^{2n+5}(X;Z_2).
\end{aligned}
$$

But $\Phi_{2n+2}(\iota^{n+1}) \in H^{2n+5}(Z,2;Z_2)/Sq^2H^{2n+3}(Z,2;Z_2) = 0$, which proves the theorem.

11. Cohomology operations in fibre spaces. In this section we show how Theorem 6.1 can be applied to the problem of computing primary cohomology operations in the total space of a fibre space when one knows the cohomology structure of the fibre and the base. We restrict ourselves to the stable range for simplicity; we will obtain information which is not contained in the spectral sequence of a fibre space.

Let $p: E \to B$ be a fibre space with fibre F. Let $i: F \to E$ be the inclusion. As in Section 3, let $\theta \in H^{n'+1}(\pi, n; \pi')$, $\phi \in H^q(\mathfrak{P}; G)$ and let $\theta'(\iota') = i^*(\phi) \in H^q(\pi', n'; G)$. Let $h \in H^n(B;\pi)$ be such that $\theta(h) = 0$ and $\tau(v) = h$, where $v \in H^{n-1}(F;\pi)$. Note that $\tau({}^1\theta(v)) = \theta(\tau(v)) = \theta(h) = 0$. We assume that $n' \leqq$ the sum of the connectives of B and F and that $q < n + n'$.

THEOREM 11.1. *Let $x \in H^{n'}(E;\pi')$ be such that $i^*(x) = {}^1\theta(v)$. Then*

$$\theta'(x) = p^*\Phi(h) \in H^q(E;G)/\theta' p^*(H^{n'}(B;\pi')).$$

Proof. It follows immediately from Lemma 6.2 that

$$\theta_p(h) = x \in H^{n'}(E;\pi')/p^*(H^{n'}(B;\pi')) + {}^1\theta(H^{n-1}(E;\pi)).$$

Our theorem now follows from Theorem 6.1.

Remark. In the spectral sequence of the fibre space $p: E \to B$, $\theta'(x)$

$= 0 \in E_2^{0,q}$ and hence also is zero in $E_\infty^{0,q}$. Under our dimensional restrictions, this implies that $\theta'(x) \in \operatorname{Im} p^*$. Theorem 11.1 gives more precise information.

12. Cup products in fibre spaces. This section is the analog of Section 11 for cup products. Using Theorem 8.1, we show that certain cup products in the total space of a fibre space are images of Massey triple products in the base space. These results were known to Hirsch [4] and Massey [6] in the case of sphere bundles.

Let $p\colon E \to B$ be a fibre space. Let $u \in H^p(B)$, $v \in H^q(B)$, $w \in H^r(B)$, and let $\tau(x) = u$, where $x \in H^{p-1}(F)$. We assume that $u \cup v = 0$; hence the element $v \otimes x \in E_2^{q,p-1} \approx H^q(B; H^{p-1}(F))$ goes into 0 under each d_r and gives an element $\{v \otimes x\} \in E_\infty^{q,p-1}$. We assume further that $v \cup w = 0$ and hence that $(v \otimes x) \cup w = 0 \in E_2^{q+r,p-1}$.

THEOREM 12.1. *There exists an element $y \in H^{p+q-1}(E)$ which projects into $\{v \otimes x\} \in E_\infty^{q,p-1}$ and such that*

$$(-1)^{q+1} y \cup p^*(w) = p^*\langle u, v, w\rangle \in H^{p+q+r-1}(E)/p^*(H^{q+r-1}(B) \cup w).$$

Proof. In order to apply Theorem 8.1, we must show that a representative of $(-1)^{q+1} u \overset{L}{\underset{p}{\cup}} v$ projects onto $\{v \otimes x\} \in E_\infty^{q,p-1}$. Let $\bar{x} \in C^{p-1}(F) = E_0^{0,p-1}$ represent x. Then $\delta\bar{x} = p^\#(\bar{u}) \in C^p(E)$ for some representative $\bar{u}$ of u. $\bar{v} \cup \bar{x} \in E_0^{q,p-1}$ represents $v \otimes x \in E_2^{q,p-1}$, and $\delta(\bar{v} \cup \bar{x}) = (-1)^q \bar{v} \cup p^\#(\bar{u})$. $u \overset{L}{\underset{p}{\cup}} v$ has as representative $(-1)^{q+1}\bar{v} \cup \bar{x} + (-1)^q p^\#(a)$, where $\delta a = \bar{u} \cup \bar{v}$. Thus in $E_0^{q,p-1}$, $\bar{v} \cup \bar{x}$ is a representative of $(-1)^{q+1} u \overset{L}{\underset{p}{\cup}} v$, and the theorem is proved.

Remark. In case F is $(p-2)$-connected, then y is defined modulo $p^*(H^{p+q-1}(B))$ and the theorem gives a sharper result.

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COMPACTNESS OF CERTAIN MAPPINGS.*[1]

By G. T. Whyburn.

1. Introduction. A mapping of one topological space onto another is *compact* provided the inverse image of each compact set in the range space is itself a compact set. When this property is present the action of the mapping is in most essential respects similar to that of a mapping on a compact domain space. In this paper our objective will be the development of conditions, in situations of interest, which will imply compactness of the mapping or properties closely related thereto and also to study the implications of these related properties.

Our spaces X and Y are always understood to be separable and metric. Other properties will be explicitly stated when they are assumed. The use of the word *mapping* in connection with a transformation $f(X) = Y$ always implies that the transformation is single valued and continuous.

2. Traces of mappings. If $f(X) = Y$ is a mapping and Y' is any subset of Y, a set X' in X which maps onto Y' under f, i. e., so that $f(X') = Y'$, is called a *trace* of Y'. We shall be concerned primarily with conditions under which certain sets have compact traces. We note that *any set in Y having a compact trace is automatically compact.* Also, a simple application of the Borel Theorem yields at once

(2.1) *If X and Y are locally compact, every compact set in Y has a compact trace if and only if each point of Y is interior to some set in Y having a compact trace or, equivalently, if each $y \in Y$ is interior to the image of some compact set in X.*

The set Y_0 of all points $y \in Y$ such that y is interior to the image of some compact set in S is open. Thus if we say that f has the *compact trace property* at such a point y, we have

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(2.2) *If X and Y are locally compact, the set Y_0 of all points in Y where f has the compact trace property is non-empty and open and its complement $F_0 = Y - Y_0$ is closed and non-dense.*

That F_0 is non-dense and hence Y_0 is non-empty follows by a simple category-type argument. For if X is represented as the union $X = \sum_1^\infty K_n$ of compact sets with $K_{n+1} \supset K_n$, then for any open set U in Y, $f(K_m) \cdot U$ must contain an open set U_0 for some m by local compactness of Y; and $U_0 \subset Y_0$ by definition of Y_0.

It results automatically that the sets $X_0 = f^{-1}(Y_0)$ and $E_0 = f^{-1}(F_0)$ are open and closed respectively and that X_0 is non-empty. If F_0 is empty, so that f has the compact trace property at each if its points, we say that it *has the compact trace property.* Equivalently, in case of locally compact X and Y, f has this property provided every compact set in Y has a compact trace. We then have

(2.3) *If X and Y are locally compact, $f \mid X_0$ is a mapping of X_0 onto Y_0 having the compact trace property.*

A property very close to the compact trace property has been used recently by P. McDougle [1] in characterizing the invariance of metrizability under certain mappings. This property (called P_2) requires that for each $y \in Y$ there exist a compact subset C_y of $f^{-1}(y)$ such that y is interior to the image of every open set in X containing C_y. Indeed for X and Y locally compact, this is equivalent to the compact trace property as we now show.

(2.4) THEOREM. *If X and Y are locally compact, a mapping $f(X) = Y$ has property P_2 if and only if every compact set in Y has a compact trace.*

Proof. Property P_2 implies that every compact set K in Y has a compact trace. For if $y \in K$, by P_2 there exists a compact subset H_y of $f^{-1}(y)$ such that if U_y is an open set containing H_y, then y is interior to $f(U_y)$. For each $y \in K$ let use choose such a U_y so that $\bar{U}_y$ is compact. Then since K is interior to $\sum f(U_y)$, there exists a finite set of points $y_1, y_2, \cdots, y_n$ in Y such that K is interior to $\sum_1^n f(U_{y_i})$. Accordingly if H denotes the compact set

$$H = f^{-1}(K) \cdot \sum_1^n \bar{U}_i,$$

$f(H) = K$ so that H is a compact trace for K.

To prove the reverse implication, let $y \in Y$ and let V be a neighborhood of y in Y with $\bar{V}$ compact. Then if A is a compact trace of $\bar{V}$ and U is any open set containing $A \cdot f^{-1}(y)$, y must be interior to $f(U)$ because $f(A) = \bar{V}$ and $f \mid A$ is a compact mapping.

We recall that a mapping $f(X) = Y$ is *quasi-open* provided that each $y \in Y$ is interior to the image $f(U)$ of every open set U in X which contains a compact component of $f^{-1}(y)$. If in addition there exists a compact component of $f^{-1}(y)$ for each $y \in Y$, we say that f is *effectively quasi-open*. Then either (2.1) or (2.4) yields at once

(2.5) *If X and Y are locally compact, every effectively quasi-open mapping $f(X) = Y$ has the compact trace property, i.e., every compact set in Y has a compact trace.*

Also a mapping $f(X) = Y$ is *monotone* provided $f^{-1}(y)$ is a continuum (compact and connected) for each $y \in Y$. For such mappings on locally compact spaces, having a compact trace is equivalent to having a compact inverse for a subset of Y. For we have in general

(2.6) THEOREM. *If X is locally compact and $f(X) = Y$ is monotone, any set in Y which has a compact trace has a compact inverse.*

For if the compact set H in Y has a compact trace K so that $f(K) = H$, $f^{-1}(H)$ must be compact. For if not we could choose a sequence of points $x_1, x_2, \cdots$ in $f^{-1}(H)$ so that $\sum x_n$ has no limit point but if $y_n = f(x_n)$, the points y_n are all distinct and the sequence (y_n) converges to a point $y \in H$. Then if U is a conditionally compact open set containing $K + f^{-1}(y)$ and C is the boundary of U, we must have $C \cdot f^{-1}(y_n) \neq 0$ for almost all n since for each n, $f^{-1}(y_n)$ is connected and intersects K and contains x_n. However, as C is compact, this gives $C \cdot \limsup f^{-1}(y_n) \neq 0$, contrary to the facts that lim-sup $f^{-1}(y_n) \subset f^{-1}(y)$ (by continuity of f) and $f^{-1}(y) \subset U$.

(2.61) COROLLARY. *If X is locally compact and $f(X) = Y$ is monotone, for any compact set K in X, $f^{-1}f(K)$ is compact. If K is a continuum, so also is $f^{-1}f(K)$.*

(2.62) COROLLARY. *A montone mapping on a locally compact space is compact if and only if it has the compact trace property.*

Note. That it is insufficient, in the theorem just proven, to have point inverses compact is shown by the mapping of the interval $0 \leqq x < 2$ onto the interval $0 \leqq y \leqq 1$ defined by $y = x$ for $0 \leqq x \leqq 1$, $y = 2 - x$ for $1 \leqq x < 2$.

Here point-inverses are either single points or point pairs and every compact set has a compact trace but of course the compact interval $0 \leqq y \leqq 1$ does not have a compact inverse.

3. Compact boundary traces. We next develop some conditions under which compactness of the mapping is a consequence of assumptions concerning the traces or inverses of certain boundary sets in the range space. A connected locally compact separable metric space is called a *generalized continuum.* We prove first

(3.1) LEMMA. *Let $f(X) = Y$ be a mapping, where X and Y are locally connected generalized continua. Let U be a conditionally compact open set in Y with boundary C such that $f^{-1}(C)$ is compact. If the number of non-conditionally compact components of $Y - C$ is finite and $\geqq$ the number of such components of $X - f^{-1}(C)$, then $f^{-1}(\bar{U})$ is compact.*

Proof. Let $R_1, R_2, \cdots, R_k$ be the non-conditionally compact components of $X - f^{-1}(C)$ and $S_1, S_2, \cdots, S_l$ the non-conditionally compact components of $Y - C$. By hypothesis $k \leqq l$. Now $H = X - \sum_1^k R_i$ must be compact. For if not, by local compactness of X and compactness of $f^{-1}(C)$, infinitely many distinct components of $X - f^{-1}(C)$, would intersect the boundary F of a conditionally compact neighborhood V of $f^{-1}(C)$ so that F would contain a point x of the limit superior of this sequence of components. This clearly is impossible by local connectedness of V at x. Hence H is compact.

However, we must have $f^{-1}(\bar{U}) \subset H$. For since $f(R_i)$ is connected, for each i, and lies in $Y - C$, each S_j must contain one of the sets $f(R_i)$ because $f(H)$ is compact. Since $l \geqq k$, we must have $f(R_i) \subset \sum S_j \subset Y - \bar{U}$ for each i so that $f^{-1}(\bar{U}) \subset H$.

(3.2) THEOREM. *Let $f(X) = Y$ be a mapping, where X and Y are locally connected generalized continua having the property that there is an integer $k \geqq 1$ such that the complement of each compact set in X or in Y has exactly k non-conditionally compact components. If for each $y \in Y$ there exist arbitrarily small neighborhoods of y whose boundaries have compact inverses under f, then f is compact.*

This is an easy consequence of the lemma. For if K is any compact set in Y, for each $y \in Y$ there exists a conditionally compact open set U_y about y with boundary C_y such that $f^{-1}(C_y)$ is compact. It results from the lemma that $f^{-1}(\bar{U}_y)$ is compact. Since by the Borel Theorem a finite union

of the sets U_y covers K, it follows that $f^{-1}(K)$ lies in a compact subset of X and hence is compact.

As shown in § 3, for monotone mappings the requirement of having a compact inverse for a set can be replaced by that of having a compact trace. Thus we get

(3.3) THEOREM. *With the same conditions on X and Y as in* (3.2), *if the mapping $f(X)=Y$ is monotone and if for each $y \in Y$ there exist arbitrarily small neighborhoods of y whose boundaries have compact traces, then f is compact.*

These two theorems have interesting consequences, particularly in the cases of mappings of a real line onto itself or of one Euclidean space onto another. In these cases the complement of *every* compact set has exactly *two* or exactly *one* non-conditionally compact component. Hence we have

(3.21) COROLLARY. *Any mapping which has compact point inverses of a line onto a line is compact.*

For in this case each point y of the line Y has arbitrarily small neighborhoods with boundaries of pairs of points.

(3.31) *Any monotone mapping of one Euclidean space onto another such that each point of the range space is contained in arbitrarily small open sets whose boundaries have compact traces is a compact mapping.*

It is clear that a similar statement could be made about monotone mappings from any locally connected generalized continuuum which is the union of an increasing sequence of compact sets with connected complements onto any other such space.

4. Mappings on Bouwer Property spaces. A space X has the *Brouwer Property* [2] provided every subset of X which is homeomorphic with an open subset of X is necessarily open in X. Now it is not true in general that when two spaces are homeomorphic, any 1-1 mapping of one onto the other is necessarily a homeomorphism. Indeed we have that

Example. There exists a 1-1 *mapping of a plane locally connected generalized continuum onto itself which is not a homeomorphism.*

Let the space X consist of the whole x-axis, the positive y-axis L and the union

$$\sum_{n=1}^{\infty} C_n + R_{-n},$$

where, for each $n > 0$, C_n is the circle of radius $\frac{1}{3}$ in the upper half plane tangent to the x-axis at the point $(n, 0)$ and R_{-n} is the vertical ray in the upper half plane originating at the point $(-n, 0)$. For each point $(x, y) \in X - L$ let us define

$$h(x, y) = (x + 1, y);$$

and let h be defined on L so as to map it (1-1) and continuously onto C_1 by sending $(0, 0)$ into $(1, 0)$ and wrapping L around C_1 clockwise, say. Clearly this mapping $h(X) = X$ meets all our requirements.

It may be noted, however, that the space X does not have the Brouwer Property. We show next that such a mapping is not possible on spaces which do have this property.

(4.1) THEOREM. *If X and Y are homeomorphic locally compact spaces having the Bouwer Property, any* 1-1 *mapping of X onto Y is a homeomorphism.*

The proof may be accomplished by showing that h is an open mapping. To this end, let U be any open set in X, let $y \in h(U)$ and let x be a point of $f^{-1}(y)$ lying in U. Next let us choose an open set V containing x such that $\bar{V}$ is compact and lies wholly in U. Then $h \mid \bar{V}$ is a homeomorphism. Accordingly, the set $h(V)$ is homeomorphic with the open subset V of X and hence also homeomorphic with some open subset of Y because X and Y are homeomorphic. Thus $h(V)$ is open in Y by the Brouwer Property. Hence y is interior to $h(U)$ and h is open since $h(V) \subset h(U)$.

(4.2) THEOREM. *Let $f(X) = Y$ be monotone, where Y is locally compact, has the Brouwer Property and is homeomorphic with the natural decomposition space of f. Then f is compact.*

The natural decomposition space of f is the space Y' whose elements are the point inverses $f^{-1}(y)$, $y \in Y$, topologized by defining a set of such elements as open provided the set union of these elements is open in Y. The mapping f has the representation

$$f(x) = h\phi(x),$$

where $\phi(X) = Y'$ is the natural mapping of this decomposition and $h(Y') = Y$ is 1-1 and continuous. By hypothesis Y' and Y are homeomorphic and have the Brouwer Property. Accordingly, by (4.1), h is a homeomorphism. Also, since f is monotone and X is locally compact, the natural decomposition of f is upper-semi continuous so that ϕ is closed. Since it

has compact point inverses, ϕ is therefore compact. Hence f is compact, being the resultant of the superposition of two compact mappings.

(4.21) COROLLARY. *Any monotone mappings of a Euclidean manifold M onto itself whose natural decomposition space is homeomorphic with M is necessarily compact.*

In particular, *any monotone mapping of a line onto a line is compact*, a fact which is also a special case of (3.21) above.

5. Monotone mappings on a plane. A deeper consequence of the theorem just proven, though not nearly so easily obtained, is

(5.1) THEOREM. *Any monotone mapping of a plane onto a plane is compact.*

Let $f(X) = Y$ be monotone, where X and Y are planes. Then if no point inverse $f^{-1}(y)$ separates the plane X, this conclusion follows at once from (4.2) together with the well known theorem of R. L. Moore [3] to the effect that the natural decomposition space of f is itself a topological plane and thus is homeomorphic with Y. Thus we have only to show that no point inverse under f can separate X.

To this end, let $N = f^{-1}(y)$, where y is any point whatever of Y. By (2.2) it follows that if A is a sufficiently large closed 2-cell on X enclosing N, $f(A)$ will contain an open 2-cell V in Y. By (2.61) the set $H = f^{-1}f(A)$ is a continuum. Let C be a circle enclosing H, let Q be the component of $X - f^{-1}f(C)$ containing H and let E be its boundary. Again by (2.61), $f^{-1}f(C)$ is a continuum so that E also is a continuum. Accordingly, if we decompose the continuum $Q + E$ into the set E and the individual points of Q and let $\phi(Q+E) = S$ be the natural mapping of this decomposition, then S is a topological sphere.

Next we let $e = \phi(E)$ and decompose S into the point e and the sets $\phi f^{-1}(y)$, for all $y \in R$, where $R = f(Q)$, and let $g(S) = K$ be the natural mapping of this decomposition. Since each of the sets $\phi f^{-1}(y)$ is a continuum, it follows [4] that K is a cactoid.

Now let us define

$$h(x) = f\phi^{-1}g^{-1}(x), \qquad x \in K - g(e).$$

Then h is a 1-1 mapping of $K - g(e)$ onto R. Further, h is compact. For if $M \subset R$ is compact, $f^{-1}(M)$ is a closed set lying in Q and thus is compact.

Thus $\phi f^{-1}(M)$ is compact as is also $g\phi f^{-1}(M)$, which is identical with $h^{-1}(M)$. Hence h must be a homeomorphism.

Since R contains V, K must contain at least one topological sphere W. Let q be a non-cut point of K on W, where $q = g(e)$ if $g(e) \in W$, and let $Z = W - q$. Then Z is an open 2-cell and thus so also is $h(Z)$. Accordingly, $h(Z)$ is open in the space Y. Whence Z must be open in K and hence is open and closed in the connected set $K - q$. This gives $Z = K - q$ so that $K = W$ and $q = g(e)$. Since K is then a topological sphere, no point inverse of f lying in Q separates X so that, in particular, N does not separate X and the proof is complete.

It may be noted that, once we have shown that $h(Z)$ is open in Y, the proof may be completed by applying (2.1) and (2.62) instead of referring back to the first paragraph of this proof. For we then have y interior to $h(Z) = f(Q) \subset f(Q + C)$, and $Q + C$ is compact.

Note 1. Combining the theorem just proven with Moore's theorem [3] we can assert the following: *Given a monotone mapping $f(X) = Y$, where X is a plane and Y is a topological space, then Y is a topological plane if and only if the mapping f is compact and has point inverses which do not separate X.*

Note 2. For non-compact monotone mappings on a plane a considerable variety of image spaces is possible. For example, the complex z plane may be mapped monotonically onto the unit circle $|w| = 1$ by the mapping $w = e^{im(z)}$, where $m(z) = 2\pi r/r + 1$, $r = |z|$.

The proof given for (5.1) suffices to give the same conclusion under a somewhat weakened hypothesis as follows

(5.2) THEOREM. *Any monotone mapping of a plane X onto a locally compact space Y which contains an open 2-cell and in which every open 2-cell is an open set, is necessarily compact.*

If the first paragraph of the proof of (5.1) is omitted and the proof concluded as indicated in the paragraph just preceeding *Note* 1, the reasoning establishes (5.2) without further alteration. Of course, it results by Moore's theorem that the space Y even here is necessarily a topological plane.

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CHARACTERISTIC CLASSES AND HOMOGENEOUS SPACES, II.*

By A. Borel and F. Hirzebruch.

Chapter VI. Applications to Todd Genera.[1]

20. Integration over the fibre in $(B_T, B_G, G/T, \rho(T,G))$.

Throughout § 20, the coefficients for cohomology are the real numbers and will not be mentioned explicitly.

20.1. Let G be a compact connected Lie group, T a maximal torus, $2m$ the dimension of G/T, $\mathcal{C}$ an invariant almost complex structure on G/T, and $a_1, \cdots, a_m$ the roots of $\mathcal{C}$ (see 12.3, 13.4). $\mathcal{C}$ defines an *orientation* of G/T, and hence also an identification of $H^{2m}(G/T)$ with $\mathbf{R}$, which will always be used in this §. In the fibering $\xi = (B_T, B_G, G/T, \rho(T,G))$ (see [2, § 20] for its definition), the integration over the fibre is a linear map of $H^*(B_T)$ in $H^*(B_G)$ or of $H^{**}(B_T)$ into $H^{**}(B_G)$ which lowers degrees by $2m$, (see § 8).

The order q of $W(G)$ is equal to the Euler number $E(G/T)$ of G/T and the latter is equal to the value of the m-th Chern class on the fundamental cycle. Therefore, considering the a_i's as elements of $H^2(B_T)$, we have by 10.8

$$(1) \qquad (a_1 \cdots a_m)[G/T] = q = \text{order } W(G),$$

where the left side denotes the value of $a_1 \cdots a_m$ on an oriented fibre.

G/T is totally non-homologous to zero in ξ for real coefficients and q is also the dimension of $H^*(G/T)$, [2, § 26]. Therefore $\rho^*(T,G)$, which will be abbreviated by π^*, is injective, and we can choose homogeneous elements $h_1, \cdots, h_q \in H^*(B_T)$ with $h_q = a_1 \cdots a_m$, whose restrictions to a fibre form a basis of $H^*(G/T)$; an element $x \in H^*(B_T)$ can then be written in one and only one way in the form

$$(2) \qquad x = \pi^*(b_1)h_1 + \cdots + \pi^*(b_q)h_q \qquad (b_i \in H^*(B_G))$$

* Received June 19, 1958.

[1] Part I of this paper appeared earlier in this Journal, Vol. 80 (1958), pp. 458-538. We refer to it for the notation and a general introduction.

and we have by 8.4(1) and (1) above that

$$x^{\natural} = q \cdot b_q. \tag{3}$$

For any $w \in W(G)$, the elements $w(h_1), \cdots, w(h_{q-1})$, $\operatorname{sgn} w \cdot h_q$ (see 2.6 for $\operatorname{sgn} w$), when restricted to a fibre, also form a base of $H^*(G/T)$. Therefore, if we apply w to (2) and use 8.4(1) again, we see that

$$(w(x))^{\natural} = \operatorname{sgn} w \cdot x^{\natural}. \tag{4}$$

20.2. Lemma. *Let $x \in H^*(B_T)$ be such that $w(x) = \operatorname{sgn} w \cdot x$ for all $w \in W(G)$. Then*

$$q \cdot x = \pi^*(x^{\natural}) \cdot a_1 \cdot \cdots \cdot a_m.$$

We may consider $H^*(B_T)$ as the ring of polynomials with real coefficients on the universal covering V_T of T. Let S_i be the symmetry to the hyperplane $a_i = 0$. Then, $S_i(x) = -x$ implies that x is zero on $a_i = 0$, and hence that x is divisible by a_i. It follows that $x = y \cdot a_1 \cdots a_m$ with $y \in H^*(B_T)$ and, in view of our assumption, invariant under $W(G)$. Therefore [2, § 26], $y = \pi^*(b)$, $b \in H^*(B_G)$, and the lemma follows from (3).

20.3. Theorem. *Let $a_1, \cdots, a_m$ be the roots of an invariant almost complex structure $\mathcal{C}$ on G/T, and $\natural$ be the integration over the fibre in $(B_T, B_G, G/T, \pi)$ with respect to the orientation defined by $\mathcal{C}$. Then for $x \in H^*(B_T)$, we have*

$$\sum_{w \in W(G)} \operatorname{sgn} w \cdot w(x) = \pi^*(x^{\natural}) \cdot a_1 \cdot \cdots \cdot a_m. \tag{5}$$

Let y be the left-hand side of (5). Then $y^{\natural} = q \cdot x^{\natural}$ by (4). On the other hand, we have $w(y) = \operatorname{sgn} w \cdot y$ for any $w \in W(G)$; therefore 20.2 shows

$$q \cdot y = \pi^*(y^{\natural}) \cdot a_1 \cdot \cdots \cdot a_m$$

which proves the theorem.

It follows from 20.3 that, if $x \in H^{**}(B_T)$, then

$$\sum_{w \in W(G)} \operatorname{sgn} w \cdot w(x) = \pi^{**}(x^{\natural}) \cdot a_1 \cdots a_m. \tag{6}$$

21. Multiplicative sequences.

21.1. In this paragraph, ξ is a bundle in which F_ξ is a compact connected n-dimensional oriented manifold, G_ξ is a group of diffeomorphisms of F_ξ, and $\hat{\xi}$ is the bundle along the fibres (7.4). Γ is a commutative ring with unit and the cohomology groups of the fibres of ξ with respect to Γ are

assumed to form a constant sheaf on B_ξ. Then $\natural = \natural_\xi$ is a map of $H^{**}(E_\xi, \Gamma)$ into $H^{**}(B_\xi, \Gamma)$ which lowers degrees by n. In particular, we have

$$a^\natural = a[F_\xi] \cdot 1 \qquad (a \in H^n(E_\xi, \Gamma)), \tag{1}$$

where 1 is the unit of $H^{**}(B_\xi, \Gamma)$.

21.2. Let $\{K_j(p_1, \cdots, p_j)\}$ be a multiplicative sequence of polynomials in indeterminates p_i, with coefficients in Γ [19, § 1]. If η is a real vector bundle, we put

$$\mathcal{K}_\eta = \sum_{j \geqq 0} K_j(p_1(\eta), \cdots, p_j(\eta)). \tag{2}$$

We have $K_j(p_1(\eta), \cdots, p_j(\eta)) \in H^{4j}(B_\eta, \Gamma)$ and $\mathcal{K}_\eta \in H^{**}(B_\eta, \Gamma)$. If η is the tangent bundle to a compact oriented differentiable manifold X, then the genus $K(X)$ of X with respect to the sequence $\{K_j\}$ is defined by $K(X) = \mathcal{K}_\eta[X]$, i.e.

$$K(X) = K_r(p_1(\eta), \cdots, p_r(\eta))[X] \in \Gamma$$

if $4r = \dim X$, and $K(X) = 0$ if $\dim X \not\equiv 0 \bmod 4$.

21.3. Definition. *Let $\hat{\xi}$ be the bundle along the fibres of ξ. The multiplicative sequence $\{K_j\}$ is said to be strictly multiplicative in ξ if and only if*

$$(\mathcal{K}_{\hat{\xi}})^\natural \in H^0(B_\xi, \Gamma). \tag{i}$$

Let $\hat{p}_i$ be the Pontrjagin classes of $\hat{\xi}$. The condition (i) is equivalent to

$$K_j(\hat{p}_1, \cdots, \hat{p}_j)^\natural = 0 \qquad (4j > n). \tag{ii}$$

The restriction of $\hat{\xi}$ to a fibre of ξ is the tangent bundle to the fibre. Therefore we have by (1) for any multiplicative sequence

$$(K_j(\hat{p}_1, \cdots, \hat{p}_j))^\natural = K(F_\xi) \cdot 1 \qquad (4j = \dim F_\xi), \tag{3}$$

and therefore $\{K_j\}$ is strictly multiplicative in ξ if and only if

$$(\mathcal{K}_{\hat{\xi}})^\natural = K(F_\xi) \cdot 1. \tag{4}$$

A multiplicative sequence is always strictly multiplicative in the product bundle, because in this case, $\hat{\xi}$ may be identified with the bundle induced from the tangent bundle to F_ξ by the projection of $E_\xi = B_\xi \times F_\xi$ onto F_ξ, and then (ii) is obviously true.

21.4. In addition to 21.1, we assume that ξ is a differentiable bundle (7.4), and that B_ξ, E_ξ are compact connected oriented manifolds, the

orientation of E_ξ being induced by those of B_ξ, F_ξ taken in this order. It follows then from the definition of the integration over the fibre (8.1) or from its equivalence with the Gysin homomorphism (8.3, remark) that

$$a[E_\xi] = a^\natural[B_\xi] \qquad (a \in H^*(E_\xi, \Gamma)).$$

Let η and η' be the tangent bundles to E_ξ and B_ξ respectively. We have an exact sequence (7.6):

$$0 \to \hat{\xi} \to \eta \to \pi^*\eta' \to 0, \qquad (\pi = \pi_\xi), \tag{5}$$

and the multiplication theorem (9.7) implies

$$p(\eta) = p(\hat{\xi}) \cdot \pi^* p(\eta') \qquad \operatorname{mod} \operatorname{Tors} H^*(E_\xi, \mathbf{Z}),$$

where $\operatorname{Tors} H^*(E_\xi, \mathbf{Z})$ is the torsion subgroup of $H^*(E_\xi, \mathbf{Z})$. By the fundamental property of multiplicative sequences [19, § 1.2], this yields

$$\mathcal{K}_\eta = \mathcal{K}_{\hat{\xi}} \cdot \mathcal{K}_{\pi^*\eta'} = \mathcal{K}_{\hat{\xi}} \cdot \pi^*(\mathcal{K}_{\eta'}), \tag{6}$$

modulo the image of $\operatorname{Tors} H^*(E_\xi, \mathbf{Z}) \otimes \Gamma$ in $H^*(E_\xi, \Gamma)$.

If $E_\xi = B_\xi \times F_\xi$, then [19, § 5.2]

$$K(E_\xi) = K(B_\xi) \cdot K(F_\xi). \tag{7}$$

More generally, we have

21.5. Proposition. *Let ξ be a differentiable bundle satisfying the assumption* 21.4 *and let $\{K_j\}$ be a multiplicative sequence of polynomials with coefficients in* Γ. *If $\{K_j\}$ is strictly multiplicative in ξ, then $K(E_\xi) = K(B_\xi) \cdot K(F_\xi)$.*

Since $H^j(E_\xi, \mathbf{Z})$ has no torsion for $j = \dim E_\xi$, we get from 21.4 and 8.2

$$K(E_\xi) = \mathcal{K}_\eta[E_\xi] = (\mathcal{K}_{\hat{\xi}} \cdot \pi^* \mathcal{K}_{\eta'})^\natural[B_\xi] = \mathcal{K}_{\eta'}(\mathcal{K}_{\hat{\xi}})^\natural[B_\xi],$$

and 21.5 follows from 21.2, 21.3(4).

21.6. We repeat briefly this discussion for the case of Chern classes. Let $\{K_j(c_1, \cdots, c_j)\}$ be a multiplicative sequence of polynomials with coefficients in Γ, in indeterminates c_i. Given a complex vector bundle η, we introduce the elements $K_j(c_1(\eta), \cdots, c_j(\eta)) \in H^{2j}(X, \Gamma)$ and put

$$\mathcal{K}_\eta = \sum_{j \geqq 0} K_j(c_1(\eta), \cdots, c_j(\eta)).$$

It is an element of $H^{**}(X, \Gamma)$. If η is the complex tangent bundle to a compact connected almost complex manifold (7.3), canonically oriented, the

genus of X with respect to $\{K_j\}$ or its "K-genus" is $K(X) = \mathcal{K}_\eta[X]$. It is equal to $K_m(c_1(\eta),\cdots,c_m(\eta))[X]$ if m is the complex dimension of X.

21.7. Let ξ be as in 21.1. Assume moreover that $\hat{\xi}$ has been endowed with a complex structure $\hat{\xi}_C$ of the type considered in 7.4, that is, defined by means of an almost complex structure of F_ξ, invariant under G_ξ, and let $\hat{c}_i$ be its Chern classes. The multiplicative sequence $\{K_j\}$ is then said to be strictly multiplicative in ξ with respect to $\hat{\xi}_C$ if one of the three following equivalent conditions is fulfilled

$$\text{(i)} \qquad (\mathcal{K}_{\hat{\xi}_C})^\natural \in H^0(B_\xi, \Gamma)$$

$$\text{(ii)} \qquad (K_j(\hat{c}_1,\cdots,\hat{c}_j))^\natural = 0, \qquad (2j > \dim F_\xi)$$

$$\text{(iii)} \qquad (\mathcal{K}_{\hat{\xi}_C})^\natural = K(F_\xi)\cdot 1.$$

21.8. Let ξ and $\hat{\xi}_C$ be as before. Assume in addition that ξ is differentiable and that B_ξ carries an almost complex structure η'_C. Then an almost complex structure η_C of E_ξ is said to be *compatible with* η'_C and $\hat{\xi}_C$ if there is an exact sequence

$$0 \to \hat{\xi}_C \to \eta_C \to \pi^*(\eta'_C) \to 0, \qquad (\pi = \pi_\xi). \tag{8}$$

Since exact sequences of vector bundles of the type (5), (8) always split, η_C always exists and is determined up to isomorphism by η'_C and $\hat{\xi}_C$. The proof of the following propostition is exactly the same as in the case of Pontrjagin classes:

21.9. PROPOSITION. *We keep the assumptions of* 21.8 *and assume moreover that the multiplicative sequence* $\{K_j\}$ *is strictly multiplicative in* ξ *with respect to* $\hat{\xi}_C$. *Then* $K(E_\xi) = K(B_\xi)\cdot K(F_\xi)$.

22. The Todd genus of certain almost complex homogeneous spaces. Throughout this paragraph, all cohomology groups will be taken with real coefficients, and all characteristic classes which occur will be regarded as real classes unless otherwise mentioned. The symbol $\boldsymbol{R}$ will be omitted in real cohomology groups.

22.1. *Notation.* Let ξ and η be complex vector bundles with the same base space: $B = B_\xi = B_\eta$. We recall that the Todd multiplicative sequence $\{T_j(c_1,\cdots,c_j)\}$ has $x(1-e^{-x})^{-1}$ as its characteristic power series [19, § 1] and define the cohomology class $\mathcal{T}(\xi,\eta) \in H^{**}(B)$ by the equation

$$\mathcal{T}(\xi,\eta) = \mathrm{ch}(\eta)\sum_{j=0}^{\infty} T_j(c_1(\xi),\cdots,c_j(\xi)),$$

where $c_i(\xi) \in H^{2i}(B)$ is the i-th Chern class of ξ and $\mathrm{ch}(\eta)$ is the Chern character of η as defined in 9.1. For $d \in H^2(B)$, set

$$\mathcal{T}(\xi, d) = e^d \cdot \sum_{j=0}^{\infty} T_j(c_1(\xi), \cdots, c_j(\xi))$$

and for $d = 0$,

$$\mathcal{T}(\xi, 0) = \mathcal{T}(\xi).$$

It is clear that $\mathcal{T}(\xi, d) = \mathcal{T}(\xi, \eta)$ if η is the complex line bundle with d as its first Chern class.

More generally, as in [19, § 1], let $\{T_j(y; c_1, \cdots, c_j)\}$ be the generalized Todd sequence. This multiplicative sequence has

$$x(1+y)/(1-e^{-x(1+y)}) - yx$$

as its characteristic power series. We define $\mathcal{T}_y(\xi) \in H^{**}(B) \otimes \mathbf{R}[y]$ by the equation

$$\mathcal{T}_y(\xi) = \sum_{j=0}^{\infty} T_j(y; c_1(\xi), \cdots, c_j(\xi)).$$

Obviously, $\mathcal{T}_0(\xi) = \mathcal{T}(\xi)$.

If B is a compact almost complex manifold and if now ξ stands for the tangential complex vector bundle of B, then we set

$$\mathcal{T}(B, \eta) = \mathcal{T}(\xi, \eta), \qquad \mathcal{T}(B, d) = \mathcal{T}(\xi, d), \qquad \mathcal{T}_y(B) = \mathcal{T}_y(\xi).$$

Moreover, in agreement with the notations of [19, §§ 10, 11, 12], the following real numbers (respectively polynomials with real coefficients) are defined

$$T(B, \eta) = \mathcal{T}(B, \eta)[B],$$

$$T(B, d) = \mathcal{T}(B, d)[B],$$

$$T_y(B) = \mathcal{T}_y(B)[B] = \sum_{p=0}^{n} T^p(B) y^p, \text{ where } n = \dim_{\mathbf{C}} B.$$

$T(B)$ denotes the Todd genus $(T(B) = T_0(B) = T(B, 0))$.

Finally, let us recall the following formal fact: If c_i is the i-th elementary symmetric function in $\gamma_1, \cdots, \gamma_n$ ($c_i = 0$ for $i > n$), then

$$\sum_{j=0}^{\infty} T_j(c_1, \cdots, c_j) = e^{c_1/2} \prod_{i=1}^{n} \gamma_i / (2 \sinh(\gamma_i/2)).$$

22.2. Let G be a compact connected Lie group and T a maximal torus of G. Let V be the universal covering of T and $a_1, \cdots, a_m$ the *positive* roots of G with respect to an ordering $\mathcal{S}$ on V^* (2.4). Let ξ be a principal G-bundle and ξ_C the complex vector bundle along the fibres (7.4) of

$(E_\xi/T, B_\xi, G/T, \pi)$ which belongs to the invariant almost complex structure on G/T having $\epsilon_1 a_1, \epsilon_2 a_2, \cdots, \epsilon_m a_m$, $(\epsilon_i = \pm 1)$, as its roots (12.3 and 13.4). We orient G/T according to this almost complex structure and denote by $\natural$ the integration over the fibre with respect to this orientation (8.1). The total Chern class of $\hat{\xi}_C$ is given by the formula (10.8):

$$c(\hat{\xi}_C) = (1 + \epsilon_1 a_1)(1 + \epsilon_2 a_2) \cdots (1 + \epsilon_m a_m). \tag{1}$$

Now let d be an arbitrary element of $H^1(T) = V^*$. We may regard d as an element of $H^2(E_\xi/T)$ under the negative transgression. Then, using (1) for the cohomology class $\mathcal{T}(\hat{\xi}_C, d)$ introduced in 22.1, we have

$$\mathcal{T}(\hat{\xi}_C, d) = e^d \cdot \prod \epsilon_j a_j/(1 - \exp(-\epsilon_j a_j)) = e^{(c_1/2)+d} \prod_{j=1}^{m} a_j/(2 \sinh(a_j/2)),$$

where $c_1 = c_1(\xi_C) = \sum_{i=1}^{m} \epsilon_i a_i$. We are going to calculate $\mathcal{T}(\hat{\xi}_C, d)^\natural$ by 20.3. In view of 8.3, this is possible since ξ is induced from the universal bundle. First observe that $\prod a_j/(2 \sinh(a_j/2))$ is invariant under the operations of the Weyl group $W(G)$ since the roots are permuted up to sign and since $x/\sinh x$ is an even function in x. Thus we obtain, after setting $b = d + \frac{1}{2} c_1(\hat{\xi}_C)$ and $a = \sum_{j=1}^{m} a_j$,

$$\epsilon_1 \epsilon_2 \cdots \epsilon_m \pi^{**}(\mathcal{T}(\hat{\xi}_C, d)^\natural) = \sum_{w \in W(G)} \operatorname{Sgn}(w) e^{w(b)} / \prod_{i=1}^{m} 2 \sinh a_i/2$$

or, in the notations of 3.2:

$$\epsilon_1 \epsilon_2 \cdots \epsilon_m \pi^{**}(\mathcal{T}(\hat{\xi}_C, d)^\natural) = E(b/2\pi(-1)^{\frac{1}{2}})/E(a/4\pi(-1)^{\frac{1}{2}}). \tag{2}$$

It follows that $\mathcal{T}(\hat{\xi}_C, d)^\natural$ vanishes if b is singular. The right side of the preceding equation is a formal power series in $d, a_1, \cdots, a_m$ (regarded as elements of $H^2(E_\xi/T)$) and, as such, is an element of $H^{**}(E_\xi/T)$. On the other hand, $d, a_1, \cdots, a_m$ are originally elements of V^*, i.e. functions on V. If b is a non-singular weight, then $E(b)/E(a/2)$ is also a function on V, namely, up to a sign, the character of a certain irreducible representation of $\tilde{G}$ (for $\tilde{G}$, see 3.3). In fact, if b is a non-singular weight, then there is a unique element $w' \in W(G)$ such that $w'(b)$ is in the positive Weyl chamber (2.7) with respect to the ordering $\mathcal{S}$; i.e., $(w'(b), a_j) > 0$ for $1 \leqq j \leqq m$, and hence $w'(b) - a/2$ is the highest weight of an irreducible representation λ of $\tilde{G}$ which is uniquely determined up to equivalence. According to 3.4, the function $E(b)/E(a/2)$ on V equals the character of λ as a function on V multiplied by $\operatorname{Sgn}(w')$. Thus we have seen that $\mathcal{T}(\xi_C, d)^\natural$ is essentially given by a character.

It is clear that $w'(b) - a/2$ is integral on the unit lattice of G if and only if d has this property. Assume now that d has the property just mentioned (in other words, that $d \in H^1(T, \mathbf{Z}) \subset H^1(T) = V^*$) and that $b = d + \frac{1}{2}c_1(\hat{\xi}_C)$ is non-singular. Then λ also defines a representation of G. The λ extension (6.5) $\lambda(\xi)$ of the principal G-bundle ξ is then defined. It is a $\boldsymbol{U}(n)$-bundle and we have

$$\mathcal{T}(\hat{\xi}_C, d)^{\natural} = \operatorname{Sgn}(w')\epsilon_1\epsilon_2 \cdots \epsilon_m \cdot \operatorname{ch}(\lambda(\xi)). \tag{3}$$

Here ch is the Chern character as defined in 9.1. See also 10.2, 10.3.

22.3. In Sections 22.3 and 22.4, we shall apply the results of 22.2 to the very special case where B_ξ is a point; then $E_\xi = G$ and $E_\xi/T = G/T$. Every element of V^* may be regarded as an element of $H^2(G/T)$. Otherwise, we keep the notations of 22.2.

The homogeneous space G/T has 2^m invariant almost complex structures belonging to the 2^m possible choices of the signs $\epsilon_1, \epsilon_2, \cdots, \epsilon_m$. Now endow G/T with the invariant almost complex structure having the roots $\epsilon_1 a_1, \epsilon_2 a_2, \cdots, \epsilon_m a_m$. Then the first Chern class of G/T is

$$c_1 = c_1(G/T) = \epsilon_1 a_1 + \epsilon_2 a_2 + \cdots + \epsilon_m a_m.$$

Since $a/2$ is a weight, $c_1/2$ is also a weight, which implies by (10.1) that the integral first Chern class $c_1(G/T) \in H^2(G/T, \mathbf{Z})$ equals 0 when reduced to coefficients mod 2. *Thus the second Stiefel-Whitney class $w_2(G/T) \in H^2(G/T, \mathbf{Z}_2)$ vanishes.*

The Pontrjagin class p_i of G/T is the i-th elementary symmetric function in the a_j^2, and vanishes for $i > 0$ (10.9). Thus for $d \in H^2(G/T)$ (see end of 22.1),

$$\begin{aligned} &\mathcal{T}(G/T, d) = \exp(d + \tfrac{1}{2}c_1) \in H^*(G/T), \\ &m! \cdot T(G/T, d) = ((c_1/2) + d)^m[G/T], \\ &2^m m! \cdot T(G/T) = c_1{}^m[G/T]. \end{aligned} \tag{4}$$

On the other hand, in our special case, $T(G/T, d) \cdot 1 = \mathcal{T}(G/T, d)^{\natural}$ and we have, by 22.2, for an element $d \in V^*$, that

$$T(G/T, d) = 0 \text{ if } d + (c_1/2) \text{ is singular,} \tag{5}$$

$$T(G/T, d) = \pm \deg(\lambda), \tag{6}$$

if d is a non-singular weight, and if λ is a suitable representation.

By Theorem 4.3, the sum $c_1 = \epsilon_1 a_1 + \cdots + \epsilon_m a_m$ is non-singular if and

only if $\epsilon_1 a_1, \cdots, \epsilon_m a_m$ is a positive system of roots of G. By 4.9 and 12.4, we get that $\epsilon_1 a_1 + \cdots + \epsilon_m a_m$ is non-singular if and only if the invariant almost complex structure on G/T with roots $\epsilon_1 a_1, \cdots, \epsilon_m a_m$ is integrable.

Putting the value 0 for d in (5), we see that the Todd genus of G/T endowed with a non-integrable invariant almost complex structure vanishes. If, however, the structure is integrable, i.e., c_1 is non-singular, then (for $d=0$) we have in the notation of 22.2 that $w'(\frac{1}{2}c_1) = w'(b) = a/2$ and $\operatorname{Sgn}(w') = \epsilon_1 \epsilon_2 \cdots \epsilon_m$. Thus λ is the trivial representation of degree 1 and the Todd genus of G/T equals $\deg(\lambda) = 1$.

22.4. Let $a_1, a_2, \cdots, a_m$ be as before the positive system of roots of G with respect to some ordering $\mathcal{S}$ and let $a = \sum_{j=1}^{m} a_j$. Choose the integrable invariant almost complex (i.e. complex) structure on G/T which has $a_1, a_2, \cdots, a_m$ as its roots ($\epsilon_i = 1$) and let G/T be oriented accordingly. An arbitrary element $b \in V^*$ can be regarded as element of $H^2(G/T)$ and then the number $\delta(b) = b^m[G/T]/m!$ is defined. δ defines a homogeneous polynomial of degree m on V^*, which vanishes if $(b, a_j) = 0$, see (4) and (5). Since a_i and a_j are not proportional for $i \neq j$ and since $\delta(a/2) = 1$ by (4), we get

$$b^m[G/T] = m! \prod_{j=1}^{m} (b, a_j)/(a/2, a_j), \qquad (b \in V^*). \tag{7}$$

Formula (7) shows that *b is singular if and only if $b^m[G/T] = 0$.*

Theorem 20.3 implies immediately that

$$b^m[G/T] = (a_1 a_2 \cdots a_m)^{-1} \sum_{w \in W(G)} \operatorname{Sgn}(w) w(b)^m, \tag{8}$$

where the right side of this equation has to be regarded as a quotient of two homogeneous polynomials of degree m on V. Assume now that b is a weight. Then b is in the positive Weyl chamber ($(b, a_j) > 0$ for $1 \leqq j \leqq m$) if and only if $b - a/2$ is in the closure of the positive Weyl chamber ($(b, a_j) \geqq 0$ for $1 \leqq j \leqq m$). By (3) and (6), we get:

If b is a weight contained in the positive Weyl chamber, then

$$b^m[G/T]/m! = T(G/T, b - a/2) = \deg(\lambda), \tag{9}$$

where λ is the irreducible representation of $\tilde{G}$ with main weight $b - a/2$, and a is the sum of the roots a_j of the invariant complex structure on G/T; i.e., a is the first Chern class of G/T endowed with this complex structure.

By (7) and (8), we get well known formulas for the degree of λ, see 3.4.

22.5. In this and the following Section, we shall use 22.2 to prove the strictly multiplicative behavior of the Todd sequence in certain fibre bundles. For this purpose, we take the value 0 for d in 22.2. Then $2b$ equals the first Chern class of the complex vector bundle $\hat{\xi}_C$ along the fibres of $(E_\xi/T, B_\xi, G/T, \pi)$. By 22.2 and 22.3, we see that the integration over the fibre (with respect to the orientation of G/T induced by $\hat{\xi}_C$) gives, when applied to $\mathfrak{T}(\hat{\xi}_C)$, either 0 or $\mathrm{ch}(\lambda(\xi))$, where λ is the trivial 1-dimensional representation, and thus $\mathrm{ch}(\lambda(\xi)) = 1$. In either case, the integration over the fibre gives only a zero-dimensional term. According to the definition in 21.7, we get:

THEOREM. *Let ξ be a principle G-bundle. Choose an invariant almost complex structure on G/T and let $\hat{\xi}_C$ be the corresponding complex vector bundle along the fibres of $(E_\xi/T, B_\xi, G/T)$. Then the Todd sequence $\{T_j\}$ is strictly multiplicative in $(E_\xi/T, B_\xi, G/T)$ with respect to $\hat{\xi}_C$.*

For later use, we reformulate our result as follows: Let Ψ be a set of roots of G which contains for each root α exactly one of the roots $\alpha, -\alpha$. Then Ψ is the set of roots of an invariant almost complex structure on G/T. Orient G/T by this structure and let $\natural(\Psi)$ be the integration over the fibre in $(E_\xi/T, B_\xi, G/T)$ with respect to that orientation. Then we have

$$(10) \qquad \Big(\prod_{\alpha\in\Psi} \alpha/(1-e^{-\alpha})\Big)^{\natural(\Psi)} = \begin{cases} 1, & \text{if } \Psi \text{ is a positive system} \\ 0, & \text{otherwise.} \end{cases}$$

Let ϵ be a map of Ψ into $\{1, -1\}$ and $s(\epsilon)$ the number of elements in Ψ which are mapped by ϵ on -1, and let $\mathrm{sgn}(\epsilon) = (-1)^{s(\epsilon)}$. We have

$$\prod_{\alpha\in\Psi} \epsilon(\alpha)\cdot\alpha/(1-e^{-\epsilon(\alpha)\cdot\alpha}) = \exp\Big(\tfrac{1}{2}\sum_{\alpha\in\Psi}\epsilon(\alpha)\cdot\alpha\Big)\cdot\prod_{\alpha\in\Psi}\alpha/(2\sinh(\alpha/2)).$$

Thus, by (10),

$$(10^*) \qquad \Big(\exp\Big(\tfrac{1}{2}\sum_{\alpha\in\Psi}\epsilon(\alpha)\cdot\alpha\Big)\cdot\prod_{\alpha\in\Psi}\alpha/(2\sinh(\alpha/2))\Big)^{\natural(\Psi)}$$

is equal to $\mathrm{sgn}(\epsilon)\cdot 1$, if $\{\epsilon(\alpha)\cdot\alpha \mid \alpha\in\Psi\}$ is a positive system, and to 0, otherwise.

We give two applications of formula (10): Let ξ be a principal G-bundle for which B_ξ is a compact *oriented* manifold. Let Ψ be a *positive* system of roots of G. Orient G/T accordingly and choose for E_ξ/T that orientation which is induced by those of B_ξ and G/T. Then, for $y \in H^*(E_\xi/T)$,

we have (see 21.4):

$$y[E_\xi/T] = y^{\natural(\Psi)}[B_\xi].$$

This fact, together with (10) and 8.2, yields for every element x of $H^*(B_\xi)$ that

$$x[B_\xi] = (\pi^*(x) \cdot \prod_{\alpha \in \Psi} \alpha/(1 - \exp(-\alpha)))[E_\xi/T], \tag{11}$$

where π is the projection of E_ξ/T on B_ξ.

22.6. Now let ξ be again an arbitrary principal G-bundle. For a closed connected subgroup U of G which contains a maximal torus T of G, consider the diagram

$$E_\xi/T \xrightarrow{\pi_\nu} E_\xi/U \xrightarrow{\pi_\mu} B_\xi. \tag{12}$$

Orient G/U. Let Θ be a system of positive roots of U. Orient U/T by the invariant complex structure having Θ as its set of roots and give G/T the orientation induced by those of G/U and U/T.

According to the above diagram, we have the fibre bundles

$$\mu = (E_\xi/U, B_\xi, G/U, \pi_\mu), \qquad \nu = (E_\xi/T, E_\xi/U, U/T, \pi_\nu),$$

$$\tilde{\xi} = (E_\xi/T, B_\xi, G/T, \pi_{\tilde{\xi}}), \qquad \eta = (G/T, G/U, U/T, \pi_\eta),$$

which satisfy the assumptions of 8.4. (The ξ of 8.4 corresponds to the $\tilde{\xi}$ here and the θ of 8.4 to η.) If we define the integrations over the fibres with respect to the orientations already chosen, then formula (2) of 8.4 holds. Thus we can infer from (10) and 8.2 the

PROPOSITION. *Let x be an arbitrary element of $H^{**}(E_\xi/U)$. Then in the foregoing notation*

$$x^{\natural\mu} = (\pi_\nu^{**}(x) \cdot \prod_{\alpha \in \Theta} \alpha/(1 - e^{-\alpha}))^{\natural\tilde{\xi}}.$$

22.7. Let U be a closed connected subgroup of G containing a maximal torus T of G and assume that G/U has been endowed with an invariant almost complex structure $\mathcal{J}$. Thus G/U is oriented. Let Ψ be the set of the roots of $\mathcal{J}$ and Θ a system of positive roots of U according to which we orient U/T. Then G/T is oriented. Let ξ be again a principal G-bundle for which we consider the diagram (12) and the four fibre bundles μ, ν, $\tilde{\xi}$, η. We wish to prove that the generalized Todd sequence (22.1) is strictly multiplicative (21.7) in μ with respect to the complex vector bundle $\hat{\mu}_C$ along the fibres arising from the given invariant almost complex structure on G/U. Let n

be the complex dimension of G/U, i. e., the number of roots in Ψ. First we observe that $\pi_\nu{}^{**}(\mathcal{T}_y(\hat{\mu}_C))$ is, in virtue of 10.8, equal to the element

$$(13) \qquad \prod_{\alpha\in\Psi}((1+y)\alpha/(1-e^{-(1+y)\alpha})-y\alpha) \in H^{**}(E_\xi/T)\otimes \boldsymbol{R}[y]$$

which goes over into

$$(13^*) \qquad \prod_{\alpha\in\Psi}(1+ye^{-\alpha})\alpha/(1-\exp(-\alpha)) \in H^{**}(E_\xi/T)\otimes \boldsymbol{R}[y]$$

if one multiplies the component of complex dimension i in (13) by $(1+y)^{n-i}$. We have to prove that $\mathcal{T}_y(\hat{\mu}_C)^{\natural\mu}$ is a zero-dimensional element of $H^{**}(B_\xi)$. In view of the proposition in 22.6 and the passage from (13) to (13*), we must prove that the element

$$\Big(\prod_{\alpha\in\Psi}(1+ye^{-\alpha})\cdot\prod_{\beta\in\Theta\cup\Psi}\beta/(1-\exp(-\beta))\Big)^{\natural\xi}$$

which is equal to

$$(14) \qquad \Big(\exp\big(\sum_{\beta\in\Theta\cup\Psi}\beta/2\big)\cdot\prod_{\alpha\in\Psi}(1+ye^{-\alpha})\cdot\prod_{\beta\in\Theta\cup\Psi}\beta/(2\sinh(\beta/2))\Big)^{\natural\xi}$$

is zero-dimensional. But this is a consequence of (10*). The set $\Theta\cup\Psi$ plays here the role of Ψ in (10*). Thus the element given in (14) equals the unit of $H^{**}(B_\xi)$ multiplied by $T_y(G/U)$, (G/U has the given almost complex structure). Here we have to use that in passing from (13) to (13*), the component of complex dimension n is not changed. Using (10*), we can obtain the value of $T_y(G/U)$. In order to formulate the final result more easily, we introduce the following definition.

Definition. *Let U be a closed subgroup of G containing a maximal torus T of G. Let Ψ be a set of complementary roots of G with respect to U which contains for each complementary root α exactly one of the roots α, $-\alpha$. Let Θ be a system of positive roots of U. Then $k^p(G/U,\Psi,\Theta)$ is defined as the number of those positive systems of roots of G which contain Θ and exactly $n-p$ roots of Ψ and thus p roots of $-\Psi$ $(0\leqq 2p\leqq 2n=\dim_{\boldsymbol{R}} G/U)$.*

Using this definition, we have, in virtue of (10*) and the fact that the element given in (14) equals $T_y(G/U)\cdot 1$, that

$$(15) \qquad T_y(G/U)=\sum_{p=0}^{n}T^p(G/U)y^p=\sum_{p=0}^{n}(-y)^p k^p(G/U,\Psi,\Theta).$$

We notice that $k^p(G/U,\Psi,\Theta)$ depends only on Ψ and not on the choice of the positive system Θ of U if Ψ is the set of roots of an invariant almost complex structure on G/U. The number $k^p(G/U,\Psi,\Theta)$ is also well defined if G/U does not admit an invariant almost complex structure.

Formula (15) states in particular that the Todd genus $T(G/U)$ equals $k^0(G/U,\Psi,\Theta)$. Thus $T(G/U)$ equals 1 if $\Psi\cup\Theta$ is a positive system of roots of G and is 0 otherwise. If the invariant almost complex structure on G/U with Ψ as the set of its roots is integrable, then $\Psi\cup\Theta$ is a positive system (13.7) and thus $T(G/U)=1$. If $T(G/U)=1$, then $\Psi\cup\Theta$ is a positive system, but $\Psi\cup-\Theta$ is also positive, since $-\Theta$ is a system of positive roots of U and $k^0(G/U,\Psi,-\Theta)=T(G/U)=1$. Therefore, $T(G/U)=1$ implies that $\Psi\cup\Theta$, $\Psi\cup-\Theta$ are positive and thus closed systems. $\Theta\cup-\Theta$ is closed, since it is the set of roots of a subgroup. Thus $T(G/U)=1$ implies that $\Psi\cup\Theta\cup-\Theta$ is closed and that the given invariant almost complex structure is integrable (12.4).

We express the results of this section in the following theorem.

22.8. THEOREM. *Let G be a compact Lie group and U a closed connected subgroup of maximal rank of G. The Todd genus of an invariant almost complex structure on G/U equals* 1 (*respectively* 0) *if the structure is integrable* (*respectively not integrable*). *With respect to a maximal torus T $(T\subset U\subset G)$, let Θ be a system of positive roots of U. Assume that G/U has been given an invariant almost complex structure $\mathfrak{G}$ and that Ψ is the set of roots of $\mathfrak{G}$. Then, letting n be the complex dimension of G/U and using the definition in* 22.7, *we have*

$$T_y(G/U)=\sum_{p=0}^{n}T^p(G/U)y^p=\sum_{p=0}^{n}(-y)^p k^p(G/U,\Psi,\Theta).$$

Let ξ be a principal G-bundle. The generalized Todd sequence

$$\{T_j(y;c_1,\cdots,c_j)\}$$

is strictly multiplicative in $(E_\xi/U,B_\xi,G/U)$ with respect to the complex vector bundle along the fibres $\hat{\xi}_C$ arising from $\mathfrak{G}$. In particular, if B_ξ is a compact almost complex differentiable manifold, if ξ is differentiable and if the differentiable manifold E_ξ/U has been endowed with an almost complex structure compatible (21.8) *with the almost complex structure of B_ξ and $\hat{\xi}_C$, then*

$$T_y(E_\xi/U)=T_y(B_\xi)\cdot T_y(G/U).$$

22.9. For $y=1$, the preceding theorem gives results on the index $\tau(G/U)$, see [19, §§ 8 and 10]. These results remain correct for an arbitrary G/U not necessarily almost complex:

Let G be a compact connected Lie group and U a closed connected subgroup of G of maximal rank. Let T be a maximal torus of U. Then,

with respect to T, let Ψ be a set of complementary roots containing for each complementary root α exactly one of the roots $\alpha, -\alpha$. Let ξ be a principal G-bundle. The element $\prod_{\alpha\in\Psi} \alpha/\mathrm{tgh}\,\alpha \in H^{**}(E_\xi/T)$ is symmetric in the α^2 $(\alpha \in \Psi)$, and thus belongs to $\pi_\nu^{**}(H^{**}(E_\xi/U))$. According to 10.7,

$$\prod_{\alpha\in\Psi} \alpha/\mathrm{tgh}\,\alpha = \pi_\nu^{**}\left(\sum_{j=0}^{\infty} L_j(p'_1, p'_2, \cdots, p'_j)\right),$$

where the p'_i are the Pontrjagin classes of the real vector bundle $\hat{\mu}$ along the fibres of $(E_\xi/U, B_\xi, G/U)$. (For the L_j, see [19, § 1].) We orient the bundle along the fibres and thus also G/U by requiring that

$$\prod_{\alpha\in\Psi} \alpha = \pi_\nu^{**}(W_{2n}),$$

where W_{2n} denotes the Euler class of $\hat{\mu}$ $(2n = \dim_R G/U)$. Then the same calculations as in 22.7 show that $\{L_j\}$ is strictly multiplicative in $(E_\xi/U, B_\xi, G/U)$ and that

$$\tau(G/U) = \sum_{p=0}^{n} (-1)^p k^p(G/U, \Psi, \Theta),$$

Θ being an arbitrary system of positive roots of U. As a consequence of the strictly multiplicative behavior, we have

$$\tau(E_\xi/U) = \tau(B_\xi) \cdot \tau(G/U),$$

in case B_ξ is a compact oriented differentiable manifold and ξ a differentiable bundle and after introducing convenient orientations.

In a similar way, under the assumptions of this section, we get by setting $y = -1$ for the Euler number $E(G/U)$ that

$$E(G/U) = \sum_{p=0}^{n} k^p(G/U, \Psi, \Theta).$$

22.10. The strictly multiplicative behavior (22.8) of the Todd sequence has certain formal consequences. We follow the notations of 22.7. Let ξ be a differentiable principal G-bundle over the compact almost complex differentiable manifold B_ξ and η a complex vector bundle over B_ξ. Consider the bundle $\mu = (E_\xi/U, B_\xi, G/U, \pi_\mu)$ and the complex vector bundle $\hat{\mu}_C$ along the fibres arising from a given invariant almost complex structure on G/U. Then endow E_ξ/U with an almost complex structure compatible with that of B_ξ and with $\hat{\mu}_C$. We have

$$\mathfrak{T}(E_\xi/U, \pi_\mu^*\eta)^{\hat{\tau}\mu} = T(G/U) \cdot \mathfrak{T}(B_\xi, \eta). \tag{16}$$

Proof. Since the complex tangent bundle of E_ξ/U is the Whitney sum of $\hat{\mu}_C$ and the complex tangent bundle of B_ξ lifted under π_μ, we obtain from the Whitney multiplication theorem (9.7) that

$$\mathfrak{T}(E_\xi/U) = \mathfrak{T}(\hat{\mu}_C)\cdot\pi_\mu{}^*\mathfrak{T}(B_\xi).$$

Thus

$$\begin{aligned}\mathfrak{T}(E_\xi/U,\pi^*{}_\mu\eta)^{\natural\mu} &= (\pi_\mu{}^*(\mathrm{ch}(\eta))\cdot\pi_\mu{}^*(\mathfrak{T}(B_\xi))\cdot\mathfrak{T}(\hat{\mu}_C))^{\natural\mu}\\ &= \mathfrak{T}(\hat{\mu}_C)^{\natural\mu}\cdot\mathfrak{T}(B_\xi,\eta) = T(G/U)\cdot\mathfrak{T}(B_\xi,\eta),\end{aligned}$$

which completes the proof.

As a consequence of (16), we get (see 21.4):

$$T(E_\xi/U,\pi_\mu{}^*\eta) = T(G/U)\cdot T(B_\xi,\eta). \tag{17}$$

There is a formula for the generalized Todd sequence which is analogous to (16) and which follows from the strictly multiplicative behavior of the generalized Todd sequence. In order to write it down, we introduce the element $\mathrm{ch}_y(\eta)$ as the element obtained from $\mathrm{ch}(\eta)$ by multiplying its component of complex dimension j with $(1+y)^j$. The element $\mathrm{ch}_y(\eta)$ was denoted by $\mathrm{t}_y(\eta)$ in [19, § 12.2]. We have

$$(\mathrm{ch}_y(\pi_\mu{}^*\eta)\cdot\mathfrak{T}_y(E_\xi/U))^{\natural\mu} = T_y(G/U)\cdot\mathrm{ch}_y(\eta)\mathfrak{T}_y(B_\xi), \tag{18}$$

which implies

$$T_y(E_\xi/U,\pi_\mu{}^*\eta) = T_y(G/U)\cdot T_y(B_\xi,\eta). \tag{19}$$

For the definition of $T_y(B_\xi,\eta)$ and $T_y(E_\xi/U,\pi_\mu{}^*\eta)$, see [19, § 12]. Compare also [19, § 14.4].

22.11. Let G be a compact connected Lie group, U a closed connected subgroup of maximal rank and T a maximal torus of U. Let d be an element of $H^1(T,\boldsymbol{Z})$ which is orthogonal to all roots of U; i.e., d is invariant under all operations of the Weyl group of U. By the canonical isomorphism of $H^1(T,\boldsymbol{Z})$ with $\mathrm{Hom}(T,\boldsymbol{U}(1))$, the element d gives rise to an homomorphism of T in $\boldsymbol{U}(1)$ which has a unique extension to an homomorphism of U in $\boldsymbol{U}(1)$, also denoted by d. Now let ξ be a principal G-bundle. We extend the principal U-bundle $(E_\xi,E_\xi/U,U)$ by the homomorphism d of U in $\boldsymbol{U}(1)$. We get a principal $\boldsymbol{U}(1)$-bundle over E_ξ/U and the associated line bundle whose first Chern class we also denote by d. Following the notations of 22.6, it is clear that $\pi_\nu{}^*(d)$ is that element of $H^2(E_\xi/T,\boldsymbol{Z})$ which is obtained from the original element $d\in H^1(T,\boldsymbol{Z})$ by the negative transgression in $(E_\xi,E_\xi/T,T)$. Therefore, we may also denote $\pi_\nu{}^*d$ by d.

Now assume moreover that G/U carries an invariant complex structure. Let Ψ be the set of roots of this structure and let Θ be a system of positive roots of U. Then, by (13.7), $\Theta \cup \Psi$ is a positive system of G, to which belongs a positive Weyl chamber. Assume furthermore that d belongs to the closure of this Weyl chamber; i.e., $(d,\alpha) \geqq 0$ for $\alpha \in \Psi$. Let λ be the representation of G with main weight d and let d_ξ be the complex vector bundle over B_ξ associated with the λ-extension of ξ. Now, as in 22.10, we make the hypothesis that B_ξ is a compact almost complex manifold and that E_ξ/U has been given an almost complex structure compatible with the almost complex structure of B_ξ and the complex vector bundle $\hat{\mu}_C$ along the fibres of E_ξ/U arising from the given invariant complex structures on G/U. We have then, using the notations of 22.6, that

$$\mathfrak{T}(E_\xi/U, d)^{\natural\mu} = \mathfrak{T}(B_\xi, d_\xi). \tag{20}$$

Proof.

$$\mathfrak{T}(E_\xi/U, d)^{\natural\mu} = (\pi_\nu{}^*(e^d \cdot \pi_\mu{}^*(\mathfrak{T}(B_\xi)) \cdot \mathfrak{T}(\hat{\mu}_C)) \cdot \prod_{\alpha \in \Theta} \alpha/(1-e^{-\alpha}))^{\natural\tilde{\xi}}$$
$$= \mathfrak{T}(B_\xi)(e^d \prod_{\alpha \in \Theta \cup \Psi} \alpha/(1-e^{-\alpha}))^{\natural\tilde{\xi}}.$$

Formula (20) follows then by applying 22.2.

Remarks. (1) Formulas (16) and (20) are closely related to Grothendieck's generalized Riemann-Roch formula (not yet published); compare also with [7b, p. 241].

(2) The representation λ induces a holomorphic map β of G/U into a complex projective space $\boldsymbol{P}_q(\boldsymbol{C})$ such that $d \in H^2(E_\xi/U)$ restricted to G/U equals $\beta^*(e^*)$, where $e^* \in H^2(\boldsymbol{P}_q(\boldsymbol{C}), \boldsymbol{Z})$ is the cohomology class dual to a hyperplane (see 14.4).

23. The A-genus of certain homogeneous spaces. Throughout this paragraph, all cohomology groups are taken with real coefficients and all characteristic classes which occur are regarded as real classes.

23.1. Let $\{\hat{A}_j(p_1, \cdots, p_j)\}$ be the multiplicative sequence of polynomials [19, § 1] with $\frac{1}{2}z^{\frac{1}{2}}/\sinh \frac{1}{2}z^{\frac{1}{2}}$ as characteristic power series. The polynomials $\hat{A}_j$ are related to the A_j introduced in [19, § 1.6] by the equation

$$A_j = 2^{4j}\hat{A}_j.$$

For a real vector bundle ξ, we define the cohomology class $\hat{\mathfrak{A}}(\xi) \in H^{**}(B_\xi)$ as follows:

$$\hat{\mathfrak{a}}(\xi) = \sum_{j=0}^{\infty} \hat{A}_j(p_1(\xi), \cdots, p_j(\xi)).$$

If ξ is the tangent bundle of a differentiable manifold X, then we set $\hat{\mathfrak{a}}(\xi) = \mathfrak{a}(X)$. The genus $\hat{A}(X)$ of a compact oriented differentiable manifold X is given by

$$\hat{A}(X) = \hat{\mathfrak{a}}(X)[X].$$

$\hat{A}(X)$ vanishes if the dimension of X is not divisible by 4. For $\dim X = 4k$, we have

$$\hat{A}(X) = \hat{A}_k(p_1(X), \cdots, p_k(X))[X],$$

and, obviously, $A(X) = 2^{4k}\hat{A}(X)$, where $A(X)$ is the genus corresponding to the power series $2z^{\frac{1}{2}}/\sinh 2z^{\frac{1}{2}}$, and which is called the A-genus of X. If X is almost complex with vanishing first Chern class, then its Todd genus equals its $\hat{A}$-genus; see 22.1 and [19, p. 15].

23.2. Let G be a compact connected Lie group, T a maximal torus of G and U a closed connected subgroup of G containing T. Choose an ordering $\mathcal{S}$ (2.4), let Θ be the set of those roots which are positive with respect to $\mathcal{S}$ and belong to U, and let Ψ be the set of positive complementary roots. Orient U/T and G/T by the invariant complex structures with root systems Θ and $\Theta \cup \Psi$ respectively. We orient G/U by its Euler class using Ψ (see 22.9). Then, in the fibre bundle $(G/T, G/U, U/T)$, the orientations of G/U and and U/T induce that of G/T. Let ξ be a principal G-bundle. We adhere now strictly to the notations given in 22.6. Consider the fibre bundle

$$\mu = (E_\xi/U, B_\xi, G/U, \pi_\mu).$$

We wish to calculate the value of $\hat{A}(G/U)$ and to investigate under which conditions the sequence $\{\hat{A}_j\}$ behaves strictly multiplicatively (21.3) in μ. Let $\hat{\mu}$ be the real vector bundle along the fibres of μ. Then, by 10.7,

$$\pi_\nu{}^{**}\hat{\mathfrak{a}}(\hat{\mu}) = \prod_{\alpha \in \Psi} \alpha/(2\sinh(\alpha/2))$$

and we get, by the proposition in 22.6, that

$$\hat{\mathfrak{a}}(\hat{\mu})^{\natural\mu} = (\prod_{\beta \in \Theta} \beta/(1 - e^{-\beta}) \prod_{\alpha \in \Psi} \alpha/(2\sinh(\alpha/2)))^{\natural\xi}.$$

If we write s for the sum of all $\alpha \in \Theta$, then

$$\hat{\mathfrak{a}}(\hat{\mu})^{\natural\mu} = (e^{\frac{1}{2}s} \prod_{\alpha \in \Theta \cup \Psi} \alpha/(2\sinh(\alpha/2)))^{\natural\xi}.$$

Let a be the sum of all roots in $\Theta \cup \Psi$. Since $\Theta \cup \Psi$ is a positive system of roots for G and since G/T is oriented by the invariant complex structure

having $\Theta \cup \Psi$ as its set of roots, we get, in virtue of 22.2, that

$$\pi_\xi^{**}(\hat{\mathfrak{A}}(\hat{\mu})^{\natural\mu}) = E(s/4\pi(-1)^{\frac{1}{2}})/E(a/4\pi(-1)^{\frac{1}{2}}). \tag{1}$$

The genus $\hat{A}(G/U)$ is given by the constant term on the right side of the preceding equation which, by 22.3 (4), is also equal to $(s^m/2^m m!)[G/T]$, where $m = \dim_C(G/T)$. Thus, by 22.4 (7),

$$\hat{A}(G/U) = \prod_{\beta \in \Theta \cup \Psi} (s,\beta)/(a,\beta). \tag{2}$$

Now assume that U is the centralizer of a toral subgroup of G and that Ψ is the set of roots of an invariant complex structure on G/U. Then $a-s$ represents the first Chern class of G/U. The element $a-s$ is orthogonal to all roots of Θ (see 14.2 and 14.8). Therefore, $(s,\beta) = (a,\beta)$ for all $\beta \in \Theta$ and thus, by (2),

$$\hat{A}(G/U) = \prod_{\beta \in \Psi} (s,\beta)/(a,\beta). \tag{3}$$

23.3. THEOREM. *Let G be a compact connected Lie group, T a maximal torus of G and U a closed connected subgroup of G containing T. Choose an ordering $\mathcal{S}$ and let Θ be the set of those roots which are positive with respect to $\mathcal{S}$ and belong to U. Let s denote the sum of all $\alpha \in \Theta$. Then the following holds:*

i) *The genus $\hat{A}(G/U)$ vanishes if and only if s is a singular element.*

ii) *If ξ is a principal G-bundle and $\hat{A}(G/U) = 0$, then the sequence $\{\hat{A}_j\}$ is strictly multiplicative in $(E_\xi/U, B_\xi, G/U)$. In particular, if B_ξ is a compact orientable differentiable manifold and ξ a differentiable bundle, then $\hat{A}(G/U) = 0$ implies that $\hat{A}(E_\xi/U) = 0$ also.*

iii) *If $\hat{A}(G/U)$ is not zero, then U is the centralizer of a toral subgroup of G*; i.e., *G/U is homogeneous algebraic* (§14). *In particular, $\hat{A}(G/U)$ vanishes if the second Betti number of G/U is zero.*

iv) *If ξ is the universal principal G-bundle and $U \neq G$, then $\{\hat{A}_j\}$ is strictly multiplicative in $(E_\xi/U, B_\xi, G/U)$ if and only if $\hat{A}(G/U) = 0$.*

Proof of i) *and* ii). The statement i) follows from formula (2). If s is singular, then $E(s/4\pi(-1)^{\frac{1}{2}}) = 0$ (see 3.2). Thus, ii) follows from i) and formula (1).

The proof of iii) will be preceded by the following lemma.

23.4. LEMMA. *If G is compact, connected, and semi-simple, if T is a maximal torus of G, and if U $(U \neq G)$ is a closed connected semi-simple*

subgroup of G which contains T, then the sum s of all roots of U which are positive with respect to a given ordering $\mathcal{S}$ on $V_T{}^*$ is singular.

Proof. It is enough to prove the lemma for the case that U is a maximal connected subgroup of G, i.e., U is not contained in a closed connected subgroup of G different from U and G. In this case, we have ([7, p. 205], compare also 10.1)

$$G/U \cong (G_1/U_1) \times (G_2/U_2) \times \cdots \times (G_k/U_k),$$

where G_i is simple, rank $U_i = \text{rank}\, G_i$, and U_i is a maximal connected subgroup of G_i. The lemma holds for G/U if it is true for at least one of the factors. Thus it suffices to prove the lemma for the case G simple and U a maximal connected subgroup of G. These spaces G/U were listed in [7, p. 219], see also 13.3. Because of i), it suffices to prove the lemma for one ordering on $V_T{}^*$, for then it is proved for all orderings on $V_T{}^*$. The proof will proceed by checking the various cases with G simple and U maximal.

If $G = \boldsymbol{B}_l$ and $U = \boldsymbol{B}_i \times \boldsymbol{D}_{l-i}$ $(0 \leqq i \leqq l-2)$, the positive roots of U with respect to a suitable ordering and a suitable maximal torus are

$$\pm x_r + x_t \; (1 \leqq r < t \leqq i), \; \pm x_r + x_t \; (i+1 \leqq r < t \leqq l), \; x_r \; (1 \leqq r \leqq l).$$

The sum s of these roots is

$$\sum_{j=1}^{i} (2j-1)x_j + 2\sum_{j=1}^{l-i} (j-1)x_{i+j}$$

which is orthogonal to the root x_{i+1}, and thus s is singular.

If $G = \boldsymbol{C}_l$ and $U = \boldsymbol{C}_i \times \boldsymbol{C}_{l-i}$, the positive roots of U with respect to a suitable ordering are (16.4)

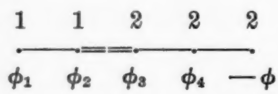
$$\pm x_r + x_t \; (1 \leqq r < t \leqq i), \; \pm x_r + x_t \; (i+1 \leqq r < t \leqq l), \; 2x_r \; (1 \leqq r \leqq l).$$

The sum s of these roots is $2\sum_{j=1}^{i} jx_j + 2\sum_{j=1}^{l-i} jx_{i+j}$ which is orthogonal to the root $-x_1 + x_{i+1}$, and thus s is singular. Next we check $\boldsymbol{F}_4/\boldsymbol{B}_4$ and $\boldsymbol{G}_2/\boldsymbol{A}_1 \times \boldsymbol{A}_1$. In these cases, one can choose a set of complementary roots containing for each complementary root α exactly one of the roots α, $-\alpha$ and such that the sum of all roots of this set is 0 (see 18.3 and 19.2). But then it is an immediate consequence of 4.4 that s is singular. The space $\boldsymbol{G}_2/\boldsymbol{A}_2$ has dimension 6 (in fact, it is the 6-sphere). Thus $\hat{A}(\boldsymbol{G}_2/\boldsymbol{A}_2) = 0$, and the lemma is correct in this case by i).

If G is simple and U maximal, one can choose orderings $\mathcal{S}$ and $\mathcal{S}'$ on $V_T{}^*$ such that each root of U simple with respect to $\mathcal{S}$ is a root of G which

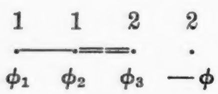
is either simple with respect to $\mathcal{S}'$ or equals the negative dominant root of G with respect to $\mathcal{S}'$ (see [7]). If $G = \boldsymbol{D}_l, \boldsymbol{E}_6, \boldsymbol{E}_7, \boldsymbol{E}_8$, then all $\mathcal{S}'$-simple roots of G and the corresponding dominant root of G have equal lengths and thus all simple roots of U with respect to the ordering $\mathcal{S}$ have all the same length ρ in the Killing metric of G. Then the sum s of all roots of U which are positive with respect to $\mathcal{S}$ has a representative contravariant vector lying in the principal diagonal of the $\mathcal{S}$-positive Weyl chamber of U [25a, p. 221], since $(s,\beta) = (\beta,\beta)$ for each $\mathcal{S}$-simple root β of U, by 3.1. According to [25a, Théorème 7], the principal diagonal contains only singular vectors, and thus s is singular.

It remains to check our lemma for the spaces $\boldsymbol{F}_4/\boldsymbol{A}_1 \times \boldsymbol{C}_3$ and $\boldsymbol{F}_4/\boldsymbol{A}_2 \times \boldsymbol{A}_2$. The Schläfli diagram of $\boldsymbol{F}_4$ (including the dominant root) is



where the ϕ_i are the $\mathcal{S}'$-simple roots of $\boldsymbol{F}_4$ and where $\phi = 2\phi_1 + 4\phi_2 + 3\phi_3 + 2\phi_4$ is the $\mathcal{S}'$-dominant root [25a]. The integers 1, 2 indicate the values of (ϕ_i, ϕ_i) and (ϕ, ϕ).

The subgroup $U = \boldsymbol{A}_1 \times \boldsymbol{C}_3$ is represented by the following diagram which indicates the $\mathcal{S}$-simple roots of U.



By an easy calculation, we get for the sum s of all $\mathcal{S}$-positive roots of U

$$s = 4\phi_1 + 6\phi_2 + 3\phi_3 - 2\phi_4$$

which is orthogonal to the root $\phi_1 + 2\phi_2 + 2\phi_3 + \phi_4$. Thus s is singular.

Finally, we consider $\boldsymbol{F}_4/\boldsymbol{A}_2 \times \boldsymbol{A}_2$. This space is almost complex (13.3) with vanishing Todd genus (22.8). Since the second Betti number of $\boldsymbol{F}_4/\boldsymbol{A}_2 \times \boldsymbol{A}_2$ is zero, also the $\hat{A}$-genus vanishes (23.1) and thus the lemma is true in this case by (i).

The proof of the lemma is completed.[2]

23.5. *Proof of* iii). The proof proceeds by induction on the dimension of G/U. Assume iii) is proved for U', G' with $\dim G'/U' < n$. We prove it now for U, G with $\dim G/U = n$. If Q is the center of G, then G/U

[2] (Added in proof). Another proof of this lemma will be given in a forthcoming paper of the authors.

$=(G/Q)/(U/Q)$ and if U/Q is the centralizer of a toral subgroup in G/Q, then the same holds for U in G, and conversely. Thus, in proving iii) for U, G with $\dim G/U = n$, we may assume that G is semi-simple. Suppose $\hat{A}(G/U) \neq 0$. Then by i) and 23.4, the subgroup U is not semi-simple. Let T' be the identity component of its center and V the centralizer of T'. Then V is connected. $V \neq G$, since G is semi-simple. Thus $U \subset V \subsetneqq G$. We have the fibre bundle $(G/U, G/V, V/U)$ and $\hat{A}(V/U) \neq 0$ by ii) . By the induction hypothesis, U is the centralizer of T' in V; but then $U = V$.

23.6. *Proof of* iv). Let ξ be the universal principal G-bundle. It follows immediately from formula (1) in 23.2 that the sequence $\{\hat{A}_j\}$ is strictly multiplicative in $(E_\xi, B_\xi, G/U)$ if and only if the function $E(s)\cdot E(a)^{-1}$, which is defined on V_T, is a constant (see 3.2). This happens in the following two cases and only then.

a) $E(s)$ is identically 0. b) $E(s) = \pm E(a)$.

We shall show that b) is impossible, if $U \neq G$. If b) holds, then s is not singular and U is the centralizer of a toral subgroup of G, according to i) and iii). From b), we infer more precisely that s is a transform of a under the Weyl group of G; i.e., $s = w(a)$ for some $w \in W(G)$. Now we can define a new ordering by letting the element $x \in V_T{}^*$ be positive if $(s, x) > 0$. Since $s = w(a)$, we conclude that s equals the sum of all roots of G which are positive in this ordering. Since $(s, \beta) > 0$ for all $\beta \in \Theta$ (notations of 23.3), all $\beta \in \Theta$ are positive in the new ordering. Since s is the sum of all roots in Θ, we infer that the sum of all complementary roots positive in the new ordering is zero which is impossible if $U \neq G$. Therefore $\{\hat{A}_j\}$ is strictly multiplicative in $(E_\xi/U, B_\xi, G/U)$ if and only if a) holds, but this is the case if and only if s is singular (3.2). In virtue of 23.3, i), the element s is singular if and only if $\hat{A}(G/U)$ vanishes. This completes the proof.

23.7. *Remarks.*

1) As a corollary of 23.3, i), we mention that s is singular if the real dimension of G/U is not divisible by 4. Thus s can only be non-singular if G/U is homogeneous algebraic of even complex dimension, see 23.3, iii).

2) In view of 23.3, ii), one might formulate the following *conjecture:* Let ξ be a bundle for which F_ξ is a compact oriented differentiable manifold and G_ξ is a group of differentiable homeomorphisms of F_ξ. If $\hat{A}(F_\xi) = 0$, then $\{\hat{A}_j\}$ is strictly multiplicative in ξ. (For the notations see 21.1-21.3.)

3) If the first Chern class of a compact almost complex manifold X vanishes, then $\hat{A}(X)$ is equal to the Todd genus of X. Taking this into account, 23.3, iii) is in agreement with 13.2 and 22.8.

4) In the proof of Lemma 23.4, we used the theorem of de Siebenthal on the principal diagonal. de Siebenthal proved his theorem by "checking all cases." There exists a general proof of it (A. Borel, unpublished).

24. Applications to simply connected algebraic homogeneous spaces.

24.1. Let X be a non-singular n-dimensional projective manifold, whose cohomology classes with respect to complex coefficients of type (p,q) vanish if $p \neq q$. Then the $h^{p,q}$ of X satisfy (see for instance [19]):

$$\chi^p(X) = \sum_{q=0}^{n} (-1)^q h^{p,q} = (-1)^p h^{p,p},$$
$$\chi^p(X) = (-1)^p \sum_{r+s=2p} h^{r,s} = (-1)^p b_{2p}, \tag{1}$$

where b_i is the Betti number of X in the real dimension i; it vanishes if i is odd. From (1) and [19, §21.3], we conclude

$$T_y(X) = \sum_{p=0}^{n} (-y)^p b_{2p}. \tag{2}$$

For $y=1$ (respectively $y=-1$), $T_y(X)$ is equal to the index $\tau(X)$ (respectively the Euler number $E(X)$) of X ([19], pp. 84, 122); hence

$$\tau(X) = \sum_{p=0}^{n} (-1)^p b_{2p}, \qquad E(X) = \sum_{p=0}^{n} b_{2p}. \tag{3}$$

24.2. Let G be a compact connected Lie group, T a maximal torus, and U the centralizer of a toral subgroup of T. Let $\mathscr{C}$ be an invariant complex structure on G/U, Ψ its root system and Θ a system of positive roots of U. Then (13.7), $\Theta \cup \Psi$ is a positive system of roots of G. The complex manifold G/U (with the structure $\mathscr{C}$) is projective and satisfies the assumptions of 24.1 (see 14.4, 14.10). It follows then from (2) and 22.7, in the notations of 22.7, that

$$b_{2p}(G/U) = k^p(G/U, \Psi, \Theta). \tag{4}$$

As was recalled in 2.7, the map $w \to w(\Theta \cup \Psi)$ is a 1-1 correspondence between the Weyl group $W(G)$ of G and the systems of positive roots. Let $W(G/U, \Psi, \Theta)$ be the set of those elements in $W(G)$ for which $\Theta \subset w(\Theta \cup \Psi)$. Each *right* coset $w(U) \cdot w$ of $W(G)$ modulo $W(U)$ contains at most one

element of $W(G/U, \Psi, \Theta)$, since only the identity of $W(U)$ transforms Θ onto Θ. Moreover, given $w \in W(G)$, the system $w(\Theta \cup \Psi)$ contains a system Θ' of positive roots of U; hence, if u is the element of $W(U)$, carrying Θ' onto Θ, we have $u \cdot w \in W(G/U, \Psi, \Theta)$. Thus $W(G/U, \Psi, \Theta)$ is a system of representatives for the right cosets of $W(G)$ modulo $W(U)$.

Given $w \in W(G)$, let $\mu(w)$ be the number of elements in $w(\Theta \cup \Psi) \cap (-\Psi)$. Then, clearly, $k^p(G/U, \Psi, \Theta)$ is the number of elements in $W(G/U, \Psi, \Theta)$ for which $\mu(w) = p$. Since Ψ is invariant under $W(U)$, (13.4, remark), we have $\mu(w) = \mu(w')$ if w and w' belong to the same right coset of $W(G)$ modulo $W(U)$. By (4) and 13.7, we have the:

24.3. THEOREM. *Let U be the centralizer of a torus in the compact connected Lie group G. Let $\mathcal{S}$ be an ordering of the roots of G for which the set Ψ of positive complementary roots is closed. For $w \in W(G)$, let $\mu(w)$ be the number of positive roots whose image under w is a negative complementary root. Then we have, with $2n = \dim G/U$:*

$$\tag{5} \begin{aligned} \sum_{p=0}^{n} b_{2p} t^{2p} &= (\operatorname{ord} W(U))^{-1} \sum_{w \in W(G)} t^{2\mu(w)}, \\ \tau(G/U) = \sum_{p=0}^{n} (-1)^p b_{2p} &= (\operatorname{ord} W(U))^{-1} \sum_{w \in W(G)} (-1)^{\mu(w)}, \end{aligned}$$

where $\tau(G/U)$ is the index of G/U, and b_{2p} its $2p$-th Betti number.

24.4. It follows in particular that $b_{2p}(G/T)$ (T maximal torus of G) equals the number of elements of $W(G)$ for which $w(\Psi)$ contains exactly p negative roots. Therefore, in the notations of 2.6, we have

$$\sum_{p=0}^{n} b_{2p}(G/T) t^{2p} = \sum_{w \in W(G)} t^{2s(w)}.$$

Theorem 24.3 was proved independently by R. Bott (Bull. Soc. Math. France 84 (1956), 251-281) in a slightly different formulation. 24.4 was also proved by C. Chevalley by means of a cellular decomposition (Tohoku Math. Journal 7 (1955), 14-66). The general case could also be read off from the cellular decomposition mentioned in [5].

24.5. *Kodaira's vanishing theorem.* Let X be a compact connected Kählerian manifold, n its complex dimension, and F a holomorphic complex line bundle over X. The bundle F is said to be negative of order $\geqq k$ if its first Chern class $c_1(F)$ can be represented by a closed real (1,1)-form ω of class C^∞ which, around every point $x \in X$, can be written in the form

$$\omega = i \sum g_{\alpha\bar{\beta}} dz_\alpha \wedge d\bar{z}_\beta$$

where $(g_{\alpha\bar\beta})$ is a hermitian matrix with at least k negative eigenvalues. In particular, F is negative of order $\geqq n$ if and only if it is negative in the sense of Kodaira, that is, if and only if F^{-1} is positive in the sense of Kodaira. In [22], Kodaira has shown that *if F is negative, then the cohomology groups $H^q(X,F)$ of X with respect to the sheaf of germs of holomorphic sections of F vanish for $q < \dim_{\mathbf{C}} X$.* By Serre's duality theorem, this is equivalent to the following statement: *Let K be the canonical bundle of X. If $F \otimes K^{-1}$ is positive, then $H^q(X,F) = 0$ for $q > 0$.*

Bott [7b, p. 231] has given a generalization of the first theorem in the case $q = 0$: *if F is negative of order $\geqq 1$, then $H^0(X,F) = 0$, that is, F does not admit a not identically zero holomorphic cross section.*

Remark. Bott formulates his theorem in a slightly different fashion; but, if one takes into account the lemma in [22, p. 1271], one gets Bott's theorem in the above form.

24.6. We keep the notations of 24.1 and 24.2. Since $H^{0,1}(G/U) = H^{0,2}(G/U) = 0$, by 14.10, the map assigning to a holomorphic line bundle over G/U its first Chern class defines an isomorphism between the group of isomorphism classes of line bundles and $H^2(G/U,\mathbf{Z})$. The negative transgression defines a homomorphism of the group A of weights which are orthogonal to the roots of U onto $H^2(G/U,\mathbf{Z})$ by 14.2. For any weight, we define $H^i(G/U,d)$ as the i-th cohomology group of G/U with respect to the sheaf of germs of holomorphic sections of a complex bundle with first Chern class d. $\chi(G/U,d)$ will denote the alternating sum of the dimensions of the $H^i(G/U,d)$.

24.7. THEOREM. *Let U be the centralizer of a torus in G, Ψ be the set of roots of an invariant complex structure on G/U, and Θ a set of positive roots for U. Let d be a weight orthogonal to the roots of U. If $(d,b) \geqq 0$ for all $b \in \Psi$, then $H^i(G/U,d) = 0$ $(i > 0)$ and $\dim_{\mathbf{C}} H^0(G/U,d)$ equals $T(G/U,d)$, which is the degree of the irreducible representation of $\tilde G$* (see 3.3) *with main weight d (in the ordering which has $\Theta \cup \Psi$ as positive roots). If $(d,b) < 0$ for at least one $b \in \Psi$, then $H^0(G/U,d) = 0$.*

The last assertion follows from Bott's theorem (24.5) and 14.6. Let c_1 be the first Chern class of G/U. Then $(c_1,b) > 0$ for $b \in \Psi$ by 14.8, and $(d,b) \geqq 0$ for $b \in \Psi$ implies $(d + c_1, b) > 0$. Since $c_1 = -c_1(K)$, the vanishing of $H^i(G/U,d)$, $(i > 0)$, follows then from 14.6 and 24.5.

Assume G/T and U/T to be endowed with the invariant complex structures having as root systems $\Theta \cup \Psi$ and Θ respectively. Then (14.3),

$(G/T, G/U, U/T, \nu)$ is a complex analytic fibering; we have by 22.8 and 22.10

$$T(G/U, d) = T(G/T, d), \tag{6}$$

and, therefore, by Riemann-Roch

$$\chi(G/U, d) = \chi(G/T, d). \tag{7}$$

Since $H^i(G/U, d) = H^i(G/T, d) = 0$ for $i > 0$, we get

$$\dim_{\mathbf{C}} H^0(G/U, d) = T(G/T, d),$$

and the remaining assertion of the theorem follows from 22.4.

Remarks. (1) Assume that $U = T$. Let $d \in H^2(G/T, \mathbf{Z})$ be such that $d + (c_1/2)$ is in the closure of the positive Weyl chamber, but not inside; therefore it is singular, $(d, b) < 0$ for at least one positive root b, and $(d + c_1, b) > 0$ for all positive roots b. By 14.6 and 24.5, it follows that *all* cohomology groups $H^i(G/T, d)$ vanish, in agreement with the fact (22.3(5)) that $T(G/T, d) = \chi(G/T, d) = 0$ if $d + (c_1/2)$ is singular.

(2) If the weight d is orthogonal to the roots of U, the element $d \in H^2(G/T, \mathbf{Z})$ is the first Chern class of a line bundle which is the image under ν^* of a line bundle on G/U with first Chern class $d \in H^2(G/U, \mathbf{Z})$. Since $H^{p,q}(U/T) = 0$ for $p \neq q$ (14.10), one can deduce by a spectral argument applied to the fibering $(G/T, G/U, U/T, \nu)$ that, more generally than in the proof of 24.7, ν^* induces an isomorphism of $H^i(G/U, d)$ onto $H^i(G/T, d)$ for all i and all d.

24.8. We assume here that G is semi-simple. Then $H^2(G/T, \mathbf{Z})$ is isomorphic to the group of weights of G.

The projective space associated to the vector space $H^0(G/T, d)$ can be identified with the complete linear system of all positive divisors whose homology class is dual to d. Thus the preceding results on $\dim H^0(G/T, d)$ are also consequences of the results of [7a] quoted in 14.4.

Bott [7b] has proved the following theorem, which had been conjectured by the authors in view of 22.2 and 24.7:

THEOREM (Bott). *Let d be a weight. Then all groups $H^i(G/T, d)$ vanish if and only if $d + (c_1/2)$ is singular. If $d + (c_1/2)$ is regular and if w is the unique element of $W(G)$ which brings $d + (c_1/2)$ into the positive Weyl chamber, then $H^i(G/T, d)$ is zero if $i \neq s(w)$, and is equal to the degree of the irreducible representation $\bar{G}$ with main weight $w(d + (c_1/2)) - (c_1/2)$ if $i = s(w)$ (see 2.6 for $s(w)$).*

24.9. *Degrees of embeddings.* We follow the preceding notations. Let d be a weight orthogonal to all roots of U for which moreover $(d,b)>0$ for all $b\in\Psi$. Let Γ be the representation with main weight d and $\check{\Gamma}$ the contragredient representation. G/U is *strictly* associated (14.4) to Γ, and $\check{\Gamma}$ induces an embedding j of G/U in the complex projective space $\boldsymbol{P}_q(\boldsymbol{C})$, where $q+1$ is the degree of the representation Γ. If $e^* = H^2(\boldsymbol{P}_q(\boldsymbol{C}),\boldsymbol{Z})$ is dual to a hyperplane of $\boldsymbol{P}_q(\boldsymbol{C})$, then $j^*(e^*)=d$ (d regarded now as element of $H^2(G/U,\boldsymbol{Z})$) and the value of the cohomology class d^n ($n=\dim_{\boldsymbol{C}} G/U$) on the fundamental cycle of G/U is the degree of the embedding in the sense of algebraic geometry. The following formula is clear for an arbitrary $d\in H^2(G/U,\boldsymbol{R})$

$$d^n[G/U] = n!\lim_{r\to\infty} r^{-n}T(G/U,rd).$$

Let a be the sum of all roots in $\Theta\cup\Psi$, then (6) and 22.3(4) and 22.4(7) give

$$T(G/U,rd) = \prod_{c\in\Theta}(rd+a/2,c)/(a/2,c)\cdot\prod_{b\in\Psi}(rd+a/2,b)/(a/2,b).$$

Since $(d,c)=0$ for $c\in\Theta$, the first product equals 1. Passing to the limit yields

$$d^n[G/U] = n!\prod_{b\in\Psi}(d,b)/(a/2,b) \qquad \text{for } d\in H^2(G/U,\boldsymbol{R}). \tag{8}$$

24.10. THEOREM. *Let G be a compact connected Lie group, T a maximal torus of G and U the centralizer of a toral subgroup of T. Endow G/U with an invariant complex structure, and let Ψ be the set of its roots. Choose an ordering $\mathcal{S}$ on V_T for which Ψ is the set of all positive complementary roots. Let a be the sum of all positive roots. Let d be a weight orthogonal to the roots of U and for which $(d,b)>0$ for all $b\in\Psi$. The contragredient representation of the irreducible representation of $\bar{G}$ (3.3) with main weight d induces an embedding of G/U in a complex projective space (14.4). The degree of this embedding in the sense of algebraic geometry is*

$$d^n[G/U] = n!\prod_{b\in\Psi}(d,b)/(a/2,b), \qquad (n=\dim_{\boldsymbol{C}} G/U). \tag{9}$$

24.11. As an example, we take $G=\boldsymbol{U}(4)$ and $U=\boldsymbol{U}(2)\times\boldsymbol{U}(1)\times\boldsymbol{U}(1)$. In 13.9, two invariant complex structures $\mathcal{C}_1$, $\mathcal{C}_2$ on G/U were defined. We shall calculate the number $c_1{}^5[G/U]$ with respect to these two structures. Let $a^{(1)}$ (respectively $a^{(2)}$) be the sum of the positive roots with respect to the ordering $\mathcal{S}_1$ (respectively $\mathcal{S}_2$) defined in 13.9. We have

$$a^{(1)} = 3x_4+x_1-x_2-3x_3, \qquad a^{(2)} = 3x_1+x_2-x_3-3x_4.$$

The first Chern classes and the roots of these two structures have been given in 13.9. With respect to the coordinates x_i, the metric in the universal covering V_T of the maximal torus of $\boldsymbol{U}(4)$ is the usual euclidean metric. Thus, in the formulas (8), (9), the scalar product is the ordinary one, and, by a straightforward computation, the Chern number $c_1{}^5[G/U]$ of G/U with respect to $\mathscr{B}_1$ (respectively $\mathscr{B}_2$) is 4860 (respectively 4500). Therefore we get an example of two 5-dimensional algebraic varieties which are C^∞-differentiably homeomorphic, but have different Chern numbers.

Chapter VII. Genera Defined by Pontrjagin Classes.

In this chapter, a real number s is said to be an integer exc 2, or integral exc 2, if there exists an integer k such that $2^k \cdot s$ is an integer. Analogously, a real cohomology class x is integral exc 2 if x, multiplied by a suitable power of 2, is the image of an integral cohomology class under the coefficient homomorphism induced by $\boldsymbol{Z} \to \boldsymbol{R}$.

25. The integrality of the A-genus.

25.1. Let $\{L_j(p_1, \cdots, p_j)\}$ and $\{A_j(p_1, \cdots, p_j)\}$ be the multiplicative sequences [19, § 1] with $z^{\frac{1}{2}}/\mathrm{tgh}\, z^{\frac{1}{2}}$ and $2z^{\frac{1}{2}}/\sinh 2z^{\frac{1}{2}}$ respectively as characteristic power series. The polynomials A_k have rational coefficients which, when written as quotients of relatively prime integers, do not contain the factor 2 in their denominators. It suffices to prove this for the coefficients a_k of the power series $2z^{\frac{1}{2}}/\sinh 2z^{\frac{1}{2}}$. The coefficient of z^k $(k \geqq 1)$ in this series is

$$a_k = (-1)^k 2^{2k+1}(2^{2k-1} - 1) B_k/(2k)\,!$$

and, by elementary number theory, $(2k)\,!$ is not divisible by 2^{2k}, whereas by von Staudt's theorem, the Bernoulli number B_k contains 2 exactly to the first power in its denominator, which proves the desired result.

25.2. If X is a compact oriented differentiable manifold, then the genera $L(X)$, $A(X)$, $\hat{A}(X)$ are defined (21.2, 23.1). They are rational numbers which vanish if the dimension of X is not divisible by 4. We have

$$A(X) = 2^{4k}\hat{A}(X) \qquad \text{for } \dim X = 4k.$$

$A(X)$ may be written with an odd denominator; by [19, Haupsatz 8.2.2], the rational number $L(X)$ equals the index $\tau(X)$ and thus is an integer. In this paragraph, we wish to prove in particular that the A-genus $A(X)$ is also an integer or, equivalently, that $\hat{A}(X)$ is integral exc 2.

For a compact almost complex manifold X with Chern classes $c_i \in H^{2i}(X, \boldsymbol{Z})$ and for elements $d_1, \cdots, d_s \in H^2(X, \boldsymbol{R})$, we define the virtual Todd genus as in [19, § 11] by the formula

$$(1) \qquad T(d_1, \cdots, d_s)_X = ((1-e^{-d_1}) \cdots (1-e^{-d_s}) \sum_{j=0}^{\infty} T_j(c_1, \cdots, c_j))[X].$$

This virtual Todd genus is a real number.

For $d \in H^2(X, \boldsymbol{R})$ the number $T(X, d)$ is defined by the formula

$$(2) \qquad T(X, d) = (e^d \sum_{j=0}^{\infty} T_j(c_1, \cdots, c_j))[X], \qquad \text{(see 22.1)}.$$

By [19, Satz 14.3.2], the Todd genus $T(X) = T(X, 0)$ is an integer exc 2 and the virtual Todd genus $T(d_1, \cdots, d_s)_X$ is integral exc 2 if $d_1, \cdots, d_s$ are images of integral cohomology classes. We have

$$T(X, d) = T(X) - T(-d)_X$$

and thus $T(X, d)$ is also integral exc 2 if d is the image of an integral class. We give now a slight generalization of these results.

25.3. PROPOSITION. *If the elements $d, d_1, \cdots, d_s$ of $H^2(X, \boldsymbol{R})$, (X compact almost complex), are integral* exc 2, *then $T(X, d)$ and the virtual Todd genus $T(d_1, \cdots, d_s)_X$ are integral* exc 2.

By (1) and (2), it is sufficient to prove that $T(X, d)$ is integral exc 2 if $2^k d$ is the image of an integral class for some positive integer k. This statement will be proved by induction on k. It is proved already for $k = 0$, and we assume it to be true for $k-1$. We have

$$e^d = (1-(1-e^{-2d}))^{-\frac{1}{2}},$$

and therefore

$$T(X, d) = \left(\sum_{r=0}^{\infty} (-1)^r \binom{-\frac{1}{2}}{r} (1-e^{-2d})^r \cdot \sum_{j=0}^{\infty} T_j(c_1, \cdots, c_j) \right) [X].$$

The coefficients $(-1)^r \binom{-\frac{1}{2}}{r} = 2^{-2r} \binom{2r}{r}$ are integers exc 2. If $2^k d$ is the image of an integral class, we see that $T(X, d)$ is a finite linear combination, with integers exc 2 as coefficients, of numbers $T(X, f)$, where f runs through certain elements of $H^2(X, \boldsymbol{R})$ for which $2^{k-1} f$ is the image of an integral class. By the induction assumption, it is therefore an integer exc 2.

The following theorem will include the integrality of the A-genus (25.2).

25.4. THEOREM. *Let X be a compact oriented differentiable manifold with the Pontrjagin classes $p_j \in H^{4j}(X, \boldsymbol{Z})$. Let the element d of $H^2(X, \boldsymbol{R})$*

be integral exc 2. *Then the number* $\hat{A}(X,d)$ *defined by*

$$\hat{A}(X,d) = (e^d \sum_{j=0}^{\infty} \hat{A}_j(p_1,\cdots,p_j))[X]$$

is integral exc 2.

The theorem is trivial if the dimension of X is odd. Therefore we may put $\dim X$ equal to $2q$. Let $\xi = (E, X, \boldsymbol{SO}(2q))$ be the principal tangent bundle of X. Let T be a maximal torus of $\boldsymbol{SO}(2q)$ and $(x_1,\cdots,x_q)$ a base of $H^1(T,\boldsymbol{Z})$, see 10.1. We consider the fibre bundle

$$\zeta = (E/T, X, \boldsymbol{SO}(2q)/T, \pi).$$

Then $\pi^*(\xi)$ is the Whitney sum of q principal $\boldsymbol{U}(1)$-bundles $\xi_1,\cdots,\xi_q$, where ξ_j is the extension of $(E, E/T, T)$ with respect to $t \to \exp 2\pi i x_j(t)$. The first Chern class of ξ_i is x_i if we regard x_i under the negative transgression of $(E, E/T, T)$ as an element of $H^2(E/T,\boldsymbol{Z})$. Let $a_1,\cdots,a_m$ $(m = q(q-1))$, be the positive roots of $\boldsymbol{SO}(2q)$ with respect to T and an ordering. The a_i are the roots of an invariant integrable almost complex structure (§ 12) on $\boldsymbol{SO}(2q)/T$, to which belongs a complex structure of the vector bundle along the fibres of ζ. Thus the principal bundle η along the fibres of ζ is restricted to $\boldsymbol{U}(m)$ and the corresponding principal $\boldsymbol{U}(m)$-bundle η' is the Whitney sum of m principal $\boldsymbol{U}(1)$-bundles $\eta_1,\eta_2,\cdots,\eta_m$ whose first Chern classes are $a_1,\cdots,a_m$ regarded as elements of $H^2(E/T,\boldsymbol{Z})$.

The principal tangent bundle of E/T is the Whitney sum of $\pi^*\xi$ and η; thus E/T admits an almost complex structure whose principal tangent $\boldsymbol{U}(m+q)$-bundle is the Whitney sum of $\eta_1,\cdots,\eta_m,\xi_1,\cdots,\xi_q$. Hence E/T is an almost complex split manifold [19, § 13.5] with total Chern class

$$c(E/T) = (1+a_1)(1+a_2)\cdots(1+a_m)(1+x_1)(1+x_2)\cdots(1+x_q).$$

By 22.5(11), we get for an arbitrary element $d \in H^2(X,\boldsymbol{R})$,

$$\hat{A}(X,d) = (\pi^*(e^d)\prod_{j=1}^{m} a_j/(1-e^{-a_j})\cdot\pi^*(\sum_{k=0}^{\infty}\hat{A}_k(p_1,\cdots,p_k)))[E/T].$$

Observing that

$$\pi^*(p(\xi)) = p(\pi^*\xi) = (1+x_1^2)(1+x_2^2)\cdots(1+x_q^2)$$

and using the identity $x/(1-e^{-x}) = (\frac{1}{2}x/\sinh\frac{1}{2}x)\cdot\exp(x/2)$, we obtain

$$\pi^*(\sum_{k=0}^{\infty}\hat{A}_k(p_1,\cdots,p_k))$$
$$= \prod_{i=1}^{q}(x_i/2)/\sinh(x_i/2) = e^{-\frac{1}{2}(x_1+\cdots+x_q)}\prod_{i=1}^{q} x_i/(1-e^{-x_i}).$$

Thus we see that

$$\hat{A}(X,d) = T(E/T, \pi^*(d) - \tfrac{1}{2}(x_1 + \cdots + x_q)),$$

and this, together with 25.3, proves 25.4.

25.5. THEOREM. *Let $\eta = (E, X, \boldsymbol{U}(k))$ be a principal bundle over a compact oriented differentiable manifold X* ($\dim X = 2q$) *and let p_j denote the Pontrjagin classes of X. Let $d \in H^2(X, \boldsymbol{R})$ be integral* exc 2. *Then the number $\hat{A}(X,d,\eta)$ defined by*

$$\hat{A}(X,d,\eta) = (e^d \operatorname{ch}(\eta) \cdot \sum_{j=0}^{\infty} \hat{A}_j(p_1, \cdots, p_j))[X]$$

is integral exc 2. (As in 9.1, $\operatorname{ch}(\eta)$ denotes the Chern character of η).

We consider the associated bundle $\zeta = (E/T, X, \boldsymbol{U}(k)/T, \pi)$ where T is the standard maximal torus of $\boldsymbol{U}(k)$. Let $(x_1, \cdots, x_k)$ be the standard base of $H^1(T, \boldsymbol{Z})$. Then $\pi^*(\eta)$ is the Whitney sum of k principal $\boldsymbol{U}(1)$-bundles whose first Chern classes are $x_1, \cdots, x_k$ if we consider $x_1, \cdots, x_k$ via negative transgression as elements of $H^2(E/T, \boldsymbol{Z})$. We note that

$$\pi^*\operatorname{ch}(\eta) = \operatorname{ch}(\pi^*\eta) = e^{x_1} + e^{x_2} + \cdots + e^{x_k}. \tag{3}$$

Let $a_1, \cdots, a_m$ $(m = k(k-1)/2)$ be the positive roots of $\boldsymbol{U}(k)$ with respect to T and an ordering. By 10.7, the Pontrjagin class $\hat{p}$ of the bundle ζ along the fibres of ζ is given by

$$\hat{p} = \prod_1^m (1 + a_i^2) \tag{4}$$

and therefore

$$\sum \hat{A}_j(\hat{p}_1, \cdots, \hat{p}_j) = \prod (a_i/2)/\sinh(a_i/2) = e^{-(a_1 + \cdots + a_m)/2} \cdot \prod a_i/(1 - e^{-a_i}). \tag{5}$$

The tangent bundle to E/T is the direct sum of $\hat{\zeta}$ and of $\pi^*\sigma$ where σ is the tangent bundle to X (7.6). Thus if p'_i denotes the i-th Pontrjagin class of E/T, we have in view of (5)

$$\pi^*(\sum \hat{A}_j(p_1, \cdots, p_j)) \cdot \prod a_i/(1 - e^{-a_i}) = e^{(a_1 + \cdots + a_m)/2} \sum \hat{A}_j(p'_1, \cdots, p'_j). \tag{6}$$

On the other hand, it follows from 22.5(11) that

$$\hat{A}(X,d,\eta) = (\pi^*(e^d \cdot \operatorname{ch}(\eta) \cdot \sum \hat{A}_j(p_1, \cdots, p_j)) \cdot \prod a_i/(1 - e^{-a_i}))[E/T].$$

Together with (3) and (6), this gives

$$\hat{A}(X,d,\eta) = \sum_i \hat{A}(E/T, \pi^*(d) + x_i + (a_1 + \cdots + a_m)/2),$$

and the right hand side is an integer exc 2 by Theorem 25.4.

25.6. *Remarks.* The preceding theorem is the most general integrality theorem we give in this paper. All the theorems of integrality for the Todd genus, etc., [19, § 14.4, 2)] are formal consequences of it: Let X be a compact almost complex manifold of complex dimension q and η a principal $\boldsymbol{U}(k)$-bundle over X. We have $T(X,\eta)=\hat{A}(X,c_1/2,\eta)$. Thus $T(X)$ and $T(X,\eta)$ are integers exc 2. As a consequence, $T_y(X)$ and $T_y(X,\eta)$ are polynomials in y with integers exc 2 as coefficients [19, p. 93, (7), (8)]. The virtual T_y-characteristic $T_y(v_1,\cdots,v_r\,|,\eta)_X$ as defined in [19, p. 95], ($v_1,\cdots,v_r$ are elements of $H^2(X,\boldsymbol{Z})$), is a polynomial of degree $q-r$ in y which can be written as a formal power series in y, the coefficients being finite linear combinations with integral coefficients of polynomials $T_y(X,\xi)$, where ξ runs through certain unitary bundles depending on η, $v_1,\cdots,v_r$. This is purely formal (see also the analogous statement for the χ_y-theory, [19, p. 132]). Thus, $T_y(v_1,\cdots,v_r\,|,\eta)_X$ is also a polynomial with integers exc 2 as coefficients.

The proofs of 25.4 and 25.5 depend mainly on the strictly multiplicative behaviour of $x(1-e^{-x})^{-1}$, and on Proposition 25.3 which we actually would need only for almost complex split manifolds. The theory of Thom enters implicitly in the proof of 25.3 (integrality of virtual indices, see [19, § 9 and end of § 13]).

We do not know how far in 25.5 "integral exc 2" could be replaced by "integral." We can only dare the following *conjectures* which are motivated by the theorem of Riemann-Roch (see [18]). Let X be a compact oriented differentiable manifold and η a principal $\boldsymbol{U}(k)$-bundle over X.

1) *Let w_2 denote the second Stiefel-Whitney class of X, ($w_2 \in H^2(X,\boldsymbol{Z}_2)$). If $d\in H^2(X,\boldsymbol{Z})$ reduced* mod 2 *is w_2, then $\hat{A}(X,d/2,\eta)$ is an integer.*

2) *If* $w_2=0$ *and* $\dim X\equiv 4 \pmod 8$, *then $\hat{A}(X)$ is an even integer.*

2*) *If* $w_2=0$, $\dim X\equiv 4 \pmod 8$ *and if the structural group of η can be reduced to $\boldsymbol{SO}(k)$, then $\hat{A}(X,0,\eta)$ is an even integer.*

These conjectures would be generalizations of Rohlin's theorem [24] that the Pontrjagin number $p_1[X]$ is divisible by 48 if $\dim X=4$ and $w_2=0$. Rohlin's theorem goes over into conjecture 2 fro $\dim X=4$.[3]

25.7. *Examples.* Putting the value 0 for d in 25.4 yields that $\hat{A}(X)$

[3] (Added in proof). A proof of (1), using the integrality of the Todd genus recently proved by Milnor (yet unpublished) will be given in the paper mentioned in footnote 2). For a different approach which proves (1) and (2*), see F. Hirzebruch, Séminaire Bourbaki, Exposé 177, Febr. 1959.

is integral exc 2 or, equivalently, that the A-genus of X (see 25.2) is an integer. This is non-trivial only if the dimension of X is divisible by 4. For $\dim X = 8$, we get [19, p. 14]

$$(5) \qquad (-4p_2 + 7p_1^2)[X] \equiv 0 \pmod{45}.$$

The integrality of the L-genus (index) gives

$$(6) \qquad (7p_2 - p_1^2)[X] \equiv 0 \pmod{45}.$$

The two congruences (5) and (6) are not independent of each other; (5) results if one multiplies (6) by -7. For $\dim X = 12$, the integrality of $A(X)$ and $L(X)$ respectively means

$$(7) \qquad (16p_3 - 44p_2p_1 + 31p_1^3)[X] \equiv 0 \pmod{945},$$

$$(8) \qquad (62p_3 - 13p_2p_1 + 2p_1^3)[X] \equiv \pmod{945}.$$

In this case, neither of the two congruences is a formal consequence of the other, since one can derive from (7) and (8) that

$$(9) \qquad (p_1p_2)[X] \equiv 0 \pmod{3}.$$

(8) is a formal consequence of (7) and (9), and (7) of (8) and (9). The congruence (9) can also be obtained by the use of Steenrod's reduced powers. In fact, by [15, Theorem 2.1],

$$p_1^3 \equiv -p_1(7p_2 - p_1^2) \pmod{3}.$$

Assume now that X is a compact connected oriented differentiable manifold of dimension $2q$, whose real Pontrjagin classes p_j vanish for $j \neq 0, \dim X$. Taking into account that $\hat{A}(X)$ is integral exc 2, we get

(10) $\quad d^q[X]/q!$ *is integral* exc 2 *for all* $d \in H^2(X, \mathbf{Z})$

and also, by 25.5,

(11) $\quad \mathrm{ch}(\eta)[X]$ *is integral* exc 2 *for every* $\boldsymbol{U}(k)$*-bundle over* X.

Let c_j be the Chern classes of η. We infer from the definition of the Chern character (9.1) that for $q \neq 0$, the $2q$-dimensional component $\mathrm{ch}(\eta)_q$ of $\mathrm{ch}\,\eta$ is of the form

$$(12) \qquad q!\,\mathrm{ch}(\eta)_q = (-1)^{q+1} q \cdot c_q + P(c_1, \cdots, c_{q-1}),$$

where P is a polynomial in $q-1$ indeterminates with integral coefficients. Therefore (11) and (12) prove in particular the following:

25.8. THEOREM. *Let ξ be a $\boldsymbol{U}(k)$-bundle over $\boldsymbol{S}_{2q}$, and let c_q be its q-th Chern class. Then $c_q[\boldsymbol{S}_{2q}]/(q-1)!$ is an integer* exc 2.

The theorem is non trivial only for $k \geqq q$. For $q=k$, it implies that the spheres $\boldsymbol{S}_{2q}$ are not almost complex for $q \geqq 4$. (See also [18, § 2.1].) This was proved by Borel-Serre by showing that $c_q[\boldsymbol{S}_{2q}]$ is divisible by every prime p less than q and not dividing q, see [6, Propositions 12.4 and 15.1].

25.9. COROLLARY. *Let η be a principal $\boldsymbol{O}(k)$-bundle ($\boldsymbol{Sp}(k)$-bundle) over the sphere $\boldsymbol{S}_{4q}$. Let p_q (respectively e_q) be the q-th Pontrjagin class (q-th symplectic Pontrjagin class) of η. Then*

$$p_q[\boldsymbol{S}_{4q}]/(2q-1)! \text{ or } e_q[\boldsymbol{S}_{4q}]/(2q-1)! \text{ respectively}$$

is an integer exc 2.

For the proof, it is enough to observe that p_q (respectively e_q) is by definition up to sign the Chern class c_{2q} of the complex extension η' of η, with respect to the inclusion $\boldsymbol{O}(k) \subset \boldsymbol{U}(k)$ (respectively $\boldsymbol{Sp}(k) \subset \boldsymbol{U}(2k)$), and then to apply Theorem 25.8 to η'.

26. Applications to homotopy groups of Lie groups. In this paragraph, C_2 will be the class of finite commutative 2-groups.

26.1. The boundary homomorphism in the homotopy sequence of a bundle ξ will be denoted by ∂_ξ. We recall that there is a commutative diagram

$$\text{(1)} \qquad \begin{array}{ccccc} \pi_i(B_\xi) & \longleftrightarrow & \pi_i(E_\xi \operatorname{mod} F) & \xrightarrow{\partial_\xi} & \pi_{i-1}(F) \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha} \\ H_i(B_\xi, \boldsymbol{Z}) & \longleftarrow & H_i(E_\xi \operatorname{mod} F, \boldsymbol{Z}) & \xrightarrow{\partial_*} & H_{i-1}(F, \boldsymbol{Z}), \end{array}$$

where F is some fibre, ∂_* the boundary homomorphism of the relative homology sequence and α the Hurewicz homomorphism. Using the bottom line of (1) to define transgression in homology and the corresponding maps

$$H^i(B_\xi, \boldsymbol{Z}) \longrightarrow H^i(E_\xi \operatorname{mod} F, \boldsymbol{Z}) \xleftarrow{\partial^*} H^{i-1}(F, \boldsymbol{Z})$$

to define the transgression τ_ξ in cohomology, we obtain readily the:

PROPOSITION. *Let $x \in \pi_i(B_\xi)$ and let $y \in H^{i-1}(F_\xi, \boldsymbol{Z})$ be transgressive. Then for any image $\tau_\xi(y) \in H^i(B_\xi, \boldsymbol{Z})$ of y by transgression, we have*

$$KI(\tau_\xi(y), \alpha(x)) = KI(y, \alpha\partial_\xi x),$$

where KI (for Kronecker index) denotes the standard pairing of homology and cohomology.

26.2. ι_n will denote a generator of $\pi_n(\boldsymbol{S}_n)$ or its image in $H_n(\boldsymbol{S}_n, \boldsymbol{Z})$, and $\iota_n{}^*$ the dual generator of $H^n(\boldsymbol{S}_n, \boldsymbol{Z})$. We recall [26, § 18.5] that if we associate to a principal G-bundle ξ over $\boldsymbol{S}_n$ the element $\partial_\xi(\iota_n) \in \pi_{n-1}(G)$, we define a 1-1 correspondence between the set of equivalence classes of principal G-bundles over $\boldsymbol{S}_n$ and $\pi_{n-1}(G)$. Also, since for any finite dimension, we may take a differentiable manifold as classifying space for G, each equivalence class may be represented by a differentiable bundle, and we shall assume our bundles to be differentiable whenever convenient. Clearly, if $\lambda\colon G \to G'$ is a homomorphism and if the G-bundle ξ is represented by α, then its λ-extension is represented by $\lambda_0(\alpha)$, where $\lambda_0\colon \pi_{n-1}(G) \to \pi_{n-1}(G')$ is induced by λ.

We shall be interested in the cases $n = 2q$, $G = \boldsymbol{U}(m)$ $(m \geqq q)$, $n = 4q$, $G = \boldsymbol{Sp}(r)$, $(r \geqq q)$, $n = 4q$, $G = \boldsymbol{SO}(s)$ $(s \geqq 2q+1)$, and shall denote by c^*_k or $c^*_k(\xi)$ (respectively e^*_k or $e^*_k(\xi)$, respectively p^*_k or $p^*_k(\xi)$) the value of the k-th Chern (respectively symplectic Pontrjagin, respectively Pontrjagin) class on ι_n, where $n = 2k$ (respectively $n = 4k$, respectively $n = 4k$). It follows directly from the definition of the characteristic classes by means of classifying spaces that $\xi \to c^*_k(\xi)$ (respectively $\xi \to e^*_k(\xi)$, respectively $\xi \to p^*_k(\xi)$) is a homomorphism of the $(n-1)$-th homotopy group $\pi_{n-1}(G)$ of the structural group into $\boldsymbol{Z}$; hence this homomorphism depends only on $\pi_{n-1}(G)$ modulo torsion. Finally, we recall that the maps

$$\pi_{2q-1}(\boldsymbol{U}(r)) \to \pi_{2q-1}(\boldsymbol{U}(s)), \pi_{4q-1}(\boldsymbol{Sp}(r)) \to \pi_{4q-1}(\boldsymbol{Sp}(s)) \quad (s \geqq r \geqq q)$$
$$\pi_{4q-1}(\boldsymbol{SO}(r)) \to \pi_{4q-1}(\boldsymbol{SO}(s)) \quad (s \geqq r \geqq 4q+1)$$

induced by the standard inclusions are isomorphisms [26, § 22.8, 25.2, 25.5] and that

$$\pi_{4q-1}(\boldsymbol{SO}(2r+1)) \to \pi_{4q-1}(\boldsymbol{SO}(2s+1)), \qquad (s \geqq r \geqq q),$$

is an isomorphism mod C_2; this last fact follows from the homotopy sequence of the fibering $\boldsymbol{SO}(2r+1)/\boldsymbol{SO}(2r-1) = \boldsymbol{W}_{4r-1}$, where $\mathbf{W}_{4r-1}$ is the manifold of unit tangent vectors to $\boldsymbol{S}_{2r}$, and from the existence of a map $\boldsymbol{S}_{4r-1} \to \boldsymbol{W}_{4r-1}$ which induces a C_2-isomorphism of $\pi_i(\boldsymbol{S}_{4r-1})$ onto $\pi_i(\boldsymbol{W}_{4r-1})$ for all $i \geqq 0$ (see [25], Chapitre IV, Prop. 2).

26.3. LEMMA. (a) *Let ξ be a principal $\boldsymbol{U}(q)$-bundle over $\boldsymbol{S}_{2q}$, and let η be the associated bundle with fibre $\boldsymbol{S}_{2q-1} = \boldsymbol{U}(q)/\boldsymbol{U}(q-1)$. Then*

$$\partial_\eta(\iota_{2q}) = \pm\, c^*_q(\xi) \cdot \iota_{2q-1}$$

(b) *Let ξ be a principal $\boldsymbol{Sp}(q)$-bundle over $\boldsymbol{S}_{4q}$, and η be the associated bundle with fibre $\boldsymbol{S}_{4q-1} = \boldsymbol{Sp}(q)/\boldsymbol{Sp}(q-1)$. Then $\partial_\eta \iota_{4q} = \pm e^*_q(\xi)\iota_{4q-1}$.*

The assertion (a) follows from the fact that $c_q(\xi)$ is the image by transgression of $\pm \iota_{2q-1}$ in η (see § 29) and from 26.1.

By definition, $e_q(\xi) = (-1)^q c_{2q}(\xi')$, where ξ' is the λ-extension of ξ under the inclusion $\lambda\colon \boldsymbol{Sp}(q) \to \boldsymbol{U}(2q)$. It is immediately seen that the pair inclusion $(\boldsymbol{Sp}(q-1), \boldsymbol{Sp}(q)) \to (\boldsymbol{U}(2q-1), \boldsymbol{U}(2q))$ induces a homeomorphism of $\boldsymbol{Sp}(q)/\boldsymbol{Sp}(q-1)$ onto $\boldsymbol{U}(2q)/\boldsymbol{U}(2q-1)$. As a consequence, the bundle $(\xi', \boldsymbol{S}_{4q-1})$ associated to ξ' is the λ-extension of η, and then (b) is implied by (a).

26.4. The result quoted at the end of 26.2 implies in particular that $\pi_{4q-1}(\boldsymbol{W}_{4q-1})$ is the direct sum of $\boldsymbol{Z}$ and of a finite 2-group, and that there exists an integer $2^{a(q)}$ such that the image of the Hurewicz homomorphism

$$\alpha\colon \pi_{4q-1}(\boldsymbol{W}_{4q-1}) \to H_{4q-1}(\boldsymbol{W}_{4q-1}, \boldsymbol{Z})$$

is generated by $2^{a(q)} \cdot j_q$, where j_q is a generator of $H_{4q-1}(\mathbf{W}_{4q-1}, \boldsymbol{Z})$.

Lemma. *Let ξ be a principal $\boldsymbol{SO}(2q+1)$-bundle over $\boldsymbol{S}_{4q}$, η be the associated bundle with fibre $\boldsymbol{W}_{4q-1} = \boldsymbol{SO}(2q+1)/\boldsymbol{SO}(2q-1)$, and let γ_q be a generator of $\pi_{4q-1}(\boldsymbol{W}_{4q-1})$mod 2-torsion. Then we have in the previous notation*

$$\partial_\eta \iota_{4q} = \pm 2^{-a(q)-1} p^*_q(\xi)\gamma_q \text{ modulo } 2\text{-}torsion.$$

Modulo 2-torsion, we have $\partial_\eta \iota_{4q} = c \cdot \gamma_q$, for some integer c, and therefore

$$\alpha\partial_\eta \iota_{4q} = 2^{a(q)} \cdot c \cdot j_q.$$

By § 30, $p_q(\xi)$ is the image by transgression in η of $\pm 2j_q^*$, where j_q^* is the generator of $H^{4q-1}(\boldsymbol{W}_{4q-1}, \boldsymbol{Z})$ dual to j_q. Hence we have by 26.1

$$\pm p^*_q(\xi) = KI(2j_q^*, \alpha\partial_\eta\iota_{4q}) = 2^{a(q)+1} \cdot c$$

which proves the lemma.

26.5. Theorem. There exists:

(a) *over $\boldsymbol{S}_{2q}$ a $\boldsymbol{U}(m)$-bundle with $c^*_q = (q-1)!$ for $m \geqq q$.*

(a*) *over $\boldsymbol{S}_{4q}$ a $\boldsymbol{SO}(n)$-bundle with $p^*_q = (2q-1)! \cdot 2$ for $n \geqq 4q$ and a $\boldsymbol{Sp}(m)$-bundle with $e^*_q = (2q-1)! \cdot 2$ for $m \geqq q$.*

(b) *over $\boldsymbol{S}_{4q}$, q even, a $\boldsymbol{SO}(n)$-bundle with $p^*_q = (2q-1)!$ for $n \geqq 4q+1$ (for $n \geqq 8$ if $q = 2$).*

(c) *over $\boldsymbol{S}_{4q}$, q odd, a $\boldsymbol{Sp}(m)$-bundle with $e^*_q = (2q-1)\,!$ for $m \geqq q$.*

(d) *over $\boldsymbol{S}_{4q}$ a $\boldsymbol{SO}(n)$-bundle with p^*_q equal, up to a power of 2, to the greatest odd factor of $(2q-1)\,!$ for $n \geqq 2q+1$.*

By 26.2 and the end remark in 9.7, it is enough to prove (a), (b), (c) for one particular value of m or n; (d) follows from (a*). The case $q=2$ in (b) will be dealt with in 26.6.

Let η be the principal $\boldsymbol{SO}(2q)$-bundle of the tangential bundle to $\boldsymbol{S}_{2q}$, and let $\lambda\colon \boldsymbol{Spin}(2q) \to \boldsymbol{SO}(2q)$ be the covering map. Since $w_2(\eta)=0$, the bundle η may be λ-restricted to a principal $\boldsymbol{Spin}(2q)$-bundle. In fact, $\rho(\lambda)\colon B_{\boldsymbol{Spin}(n)} \to B_{\boldsymbol{SO}(n)}$ is (for any $n \geqq 2$) a fibre map with fibre $B_{\boldsymbol{Z}_2}$ (see [2] § 22 or [6] § 1), i.e. an Eilenberg-MacLane space $K(\boldsymbol{Z}_2, 1)$; its spectral sequence shows readily that the obstruction to a cross section is the universal second Stiefel-Whitney class w_2; then, by a standard argument, every map $\sigma\colon B \to B_{\boldsymbol{SO}(n)}$ with $\sigma^*(w_2)=0$ can be factorized through $\rho(\lambda)$, and this shows in our case the existence of a λ-restriction η' of η.

Let x_i, $(1 \leqq i \leqq q)$, be the standard basis of the usual maximal torus $\boldsymbol{T}$ of $\boldsymbol{SO}(2q)$, and let $\boldsymbol{T}'$ be the inverse image of $\boldsymbol{T}$ in $\boldsymbol{Spin}(2q)$; it is connected (see 10.1) and is a maximal torus of $\boldsymbol{Spin}(2q)$; we shall also denote by x_i the image of x_i in $H^1(\boldsymbol{T}', \boldsymbol{Z})$ under the covering map; these generate a subgroup of index 2 of $H^1(\boldsymbol{T}', \boldsymbol{Z})$. Let $\beta\colon \boldsymbol{Spin}(2q) \to \boldsymbol{U}(2^{q-1})$ be one of the half-spinor representations, say the one with the highest weight $\frac{1}{2}(x_1 + \cdots + x_q)$, and let θ be the β-extension of η'). We want to prove

$$(2) \qquad c_q(\theta) = (-1)^{q-1}(q-1)\,! \cdot \iota_{2q}$$

which,[4] in view of our initial remark, will prove (a). Let ω_j $(1 \leqq j \leqq 2^{q-1})$, be the weights of β. It is known that these are just the linear forms

$$\tfrac{1}{2}(\epsilon_1 x_1 + \cdots + \epsilon_q x_q), \quad (\epsilon_i = \pm 1, (i=1,\cdots,q), \textstyle\prod \epsilon_i = 1)$$

(in fact, these are all transforms under the Weyl group $W(\boldsymbol{SO}(2q))$ of the highest weight, hence they must be weights; moreover, since there are 2^{q-1} of them, they represent all weights). Let ρ be the projection of $E_{\eta'}/T'$ onto $\boldsymbol{S}_{2q}$. By (9.5), we have $\rho^*(W_{2q}(\eta)) = x_1 \cdots x_q$, and hence

$$(3) \qquad x_1 \cdots x_q = 2 \cdot \rho^*(\iota_{2q}).$$

By (10.3), $\rho^*(c_q(\theta))$ is the q-th elementary symmetric function in the ω_j. Since the lower symmetric functions are zero here (because $H^i(\boldsymbol{S}_{2q}, \boldsymbol{Z}) = 0$

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[4] The other half-spinor representation yields a bundle whose q-th Chern class is $-c_q(\theta)$.

for $0 < i < 2q$), we get

$$(4) \qquad (-1)^{q-1} q \cdot \rho^*(c_q(\theta)) = \omega_1{}^q + \cdots + \omega_s{}^q, \qquad (s = 2^{q-1}).$$

Let $\omega_j = \frac{1}{2}(\epsilon_1 x_1 + \cdots + \epsilon_q x_q)$ be one particular weight. We have

$$\omega_j{}^q = q! \cdot 2^{-q} \cdot x_1 \cdots x_q + b_j,$$

where b_j is a sum of monomials in the x_i's, none of which contains all variables x_i, and therefore

$$\sum \omega_j{}^q = q! \cdot 2^{-1} \cdot x_1 \cdots x_q + \sum b_j$$

or, taking (3) into account,

$$\sum \omega_j{}^q = q! \cdot \rho^*(\iota_{2q}) + \sum b_j$$

so that (2) will follow from (4) if we show that

$$(5) \qquad \sum b_j = 0.$$

$W(\boldsymbol{SO}(2q))$ is the group of permutations of the x_i combined with an even number of changes of signs. Thus, the ring I_W of invariants of $W(\boldsymbol{SO}(2q))$ is generated by $x_1 \cdots x_q$ and by the symmetric functions in the x_i^2. The Weyl group permutes the ω_j, and therefore $\sum b_j \in I_W$; since no monomial in this sum contains all variables x_i's, b_j must then be a symmetric function in the x_i^2; but, by (9.3), it is then the image under ρ^* of a polynomial in the Pontrjagin classes of η. Since the Pontrjagin classes of $\boldsymbol{S}_{2q}$ are all zero, this proves (5).

Let now $q = 2s$ be even; let ξ be the $\boldsymbol{U}(2s)$-bundle over $\boldsymbol{S}_{4s}$ with Chern class $(2s-1)!$, and ξ^* be its extension under the contragredient representation (10.6). We have $c_{2s}(\xi) = c_{2s}(\xi^*)$, hence

$$c_{2s}(\xi \oplus \xi^*) = (2s-1)! \cdot 2 \cdot \iota_{4s},$$

but in $\boldsymbol{U}(4s)$, the matrices of the form $A \dotplus \bar{A}$ $(A \in \boldsymbol{U}(2s))$ form a subgroup conjugate to a subgroup of $\boldsymbol{Sp}(2s)$ or of $\boldsymbol{SO}(4s)$, and $\xi \oplus \xi^*$ can be considered as the complexification of a $\boldsymbol{Sp}(2s)$- or of a $\boldsymbol{SO}(4s)$-bundle. This proves (a*).

It is known that the image group of a half-spinor representation of $\boldsymbol{SO}(4q)$ is conjugate to a subgroup of $\boldsymbol{SO}(2^{2q-1})$ (respectively $\boldsymbol{Sp}(2^{2q-2})$), if q is even (respectively odd). (See E. Cartan, Jour. Math. Pur. Appl. 10 (1914), 149-186, § XV, p. 173, or A. I. Malcev, Isv. Ak. Nauk. SSSR Ser. Math. 8 (1944), 143-174, A. M. S. Translation 33, pp. 29-30.) This implies that θ is the complexification of a $\boldsymbol{SO}(2^{2q-1})$- (respectively $\boldsymbol{Sp}(2^{2q-2})$-) bundle, for q even (respectively odd), and (b) and (c) follow from (2).

26.6. *Remark.* The image group of $\boldsymbol{Spin}(8)$ under a half-spinor representation is conjugate to $\boldsymbol{SO}(8)$, as is well known and follows also from the result just quoted. The standard representation of $\boldsymbol{SO}(8)$ and the two half-spinor representations provide three homomorphisms of $\boldsymbol{Spin}(8)$ onto $\boldsymbol{SO}(8)$ which are, up to equivalence, all the representations of degree 8 of $\boldsymbol{Spin}(8)$. They may be distinguished by the element or order 2 of the center of $\boldsymbol{Spin}(8)$, (isomorphic to $\boldsymbol{Z}_2+\boldsymbol{Z}_2$), which they map onto the identity. They may be obtained from one another by performing on $\boldsymbol{Spin}(8)$ the automorphisms of the triality principle, (which are transitive on the elements of the center of $\boldsymbol{Spin}(8)$ different from the identity).

The last step of the proof of 26.5 shows therefore that θ is a $\boldsymbol{SO}(8)$-bundle over $\boldsymbol{S}_8$ with $p^*_2=-6$, which ends the proof of (b). On the other hand, the projective lines on the Cayley plane $\boldsymbol{W}$ are homeomorphic to $\boldsymbol{S}_8$ and a generator u of $H^8(\boldsymbol{W},\boldsymbol{Z})$ restricts on them to a fundamental cocycle; therefore 9.7 and 19.4 show that the normal bundle θ' to a projective line in $\boldsymbol{W}$ has also $p^*_2=-6$. In fact, it can be shown directly that θ and θ' are isomorphic; we sketch the proof, using 19.1 and some information on $\boldsymbol{W}$ to be found for instance in [1]: Let $\boldsymbol{U}$ be a subgroup of $\boldsymbol{F}_4$ isomorphic to $\boldsymbol{Spin}(9)$, $\boldsymbol{V}$ a subgroup of $\boldsymbol{U}$ isomorphic to $\boldsymbol{Spin}(8)$, and P the point of $\boldsymbol{W}$ fixed under $\boldsymbol{U}$. Then $\boldsymbol{V}$ leaves exactly two other points Q, R fixed, and the projective line M joining Q, R is operated upon transitively by $\boldsymbol{U}$. Thus the fibering $\xi=(\boldsymbol{U},\boldsymbol{U}/\boldsymbol{V},\boldsymbol{V})$ may be identified with $(\boldsymbol{Spin}(9), \boldsymbol{S}_8, \boldsymbol{Spin}(8))$. The natural representation of $\boldsymbol{V}$ into the tangent space $\boldsymbol{W}_Q$ of $\boldsymbol{W}$ at Q decomposes into the representations ρ_1, ρ_2 into M_Q and into the subspace N_Q of $\boldsymbol{W}_Q$ orthogonal to M_Q. Thus the tangent (respectively normal) bundle to M is the ρ_1- (respectively ρ_2-) extension of ξ. The representation of $\boldsymbol{V}$ in $\boldsymbol{W}_Q$ is faithful, because its kernel belongs to the center of $\boldsymbol{F}_4$, which is reduced to $\{e\}$. Hence (see beginning of this section) ρ_1 and ρ_2 are not equivalent, the normal bundle to M and the bundle θ of 26.5 arise from the tangent bundle to $\boldsymbol{S}_8$ by the same construction.

26.7. THEOREM. *The fibrations*

$$\boldsymbol{U}(q)/\boldsymbol{U}(q-1)=\boldsymbol{S}_{2q-1}, \qquad \boldsymbol{Sp}(q)/\boldsymbol{Sp}(q-1)=\boldsymbol{S}_{4q-1},$$

$$\boldsymbol{SO}(2q+1)/\boldsymbol{SO}(2q-1)=\boldsymbol{W}_{4q-1}, \qquad \boldsymbol{G}_2/\boldsymbol{Sp}(1)=\boldsymbol{W}_{11}$$

give rise to the following sequences, which are exact modulo the class C_2 of finite commutative 2-groups:

(a) $\quad 0\to \boldsymbol{Z}_{(q-1)!}\to \pi_{2q-2}(\boldsymbol{U}(q-1))\to \pi_{2q-2}(\boldsymbol{U}(q))\to 0 \qquad (q\geqq 2)$

(b) $0 \to \boldsymbol{Z}_{(2q-1)!} \to \pi_{4q-2}(\boldsymbol{Sp}(q-1)) \to \pi_{4q-2}(\boldsymbol{Sp}(q)) \to 0 \qquad (q \geqq 2)$

(c) $0 \to \boldsymbol{Z}_{(2q-1)!} \to \pi_{4q-2}(\boldsymbol{SO}(2q-1)) \to \pi_{4q-2}(\boldsymbol{SO}(2q+1)) \to 0 \qquad (q \geqq 2)$

(d) $0 \to \boldsymbol{Z}_{k15} \to \pi_{10}(\boldsymbol{S}_8) \to \pi_{10}(\boldsymbol{G}_2) \to 0$, *for some* $k \geqq 1$.[5]

(a) Let $\alpha \in \pi_{2q-1}(\boldsymbol{U}(q))$. Let ξ be a principal $\boldsymbol{U}(q)$-bundle over $\boldsymbol{S}_{2q}$ representing α, and η be the associated bundle with fibre $\boldsymbol{S}_{2q-1}$. Then $E_\eta = E_\xi/\boldsymbol{U}(q-1)$ and the restriction of the natural map $E_\xi \to E_\eta$ to a fibre is the projection map in the fibering $(\boldsymbol{U}(q), \boldsymbol{S}_{2q-1}, \boldsymbol{U}(q-1))$ hence we get a commutative diagram

$$\begin{array}{ccc} \pi_{2q}(\boldsymbol{S}_{2q}) & \xrightarrow{\partial_\eta} & \pi_{2q-1}(\boldsymbol{S}_{2q-1}) \\ \updownarrow & & \uparrow \psi \\ \pi_{2q}(\boldsymbol{S}_{2q}) & \xrightarrow{\partial_\xi} & \pi_{2q-1}(\boldsymbol{U}(q)), \end{array}$$

where ψ is part of the homotopy sequence of $(\boldsymbol{U}(q), \boldsymbol{S}_{2q-1}, \boldsymbol{U}(q-1))$. By definition, $\alpha = \partial_\xi(\iota_{2q})$, hence we have by 26.3a.

$$\psi(\alpha) = \pm c^*{}_q(\xi)\iota_{2q-1}$$

which is divisible by the greatest odd factor of $(q-1)!$ in virtue of 25.8. Since α is arbitrary, this shows that $\psi(\pi_{2q-1}(\boldsymbol{U}(q))$ is contained in the subgroup generated by $b(q-1)\cdot\iota_{2q-1}$, where $b(q-1)$ is the greatest odd factor of $(q-1)!$; on the other hand, the same argument together with (26.5), shows that $\psi(\pi_{2q-1}(\boldsymbol{U}(q)))$ contains $(q-1)!\cdot\iota_{2q-1}$, and the mod C_2 exactness of (a) follows.

The proofs for (b) and (c) are quite analogous, the sole difference being that one has to invoke 26.3b and 26.4 instead of 26.3a.

For the fibration $\boldsymbol{G}_2/\boldsymbol{Sp}(1) = \boldsymbol{W}_{11}$, we refer to [4, § 17]. Let $\alpha \in \pi_{11}(\boldsymbol{G}_2)$, let ξ be a principal $\boldsymbol{G}_2$-bundle representing α, and let η be the associated bundle with fibre $\boldsymbol{W}_{11}$. We have the commutative diagram

$$(6) \qquad \begin{array}{ccc} \pi_{12}(\boldsymbol{S}_{12}) & \xrightarrow{\partial_\eta} & \pi_{11}(\boldsymbol{W}_{11}) \\ \updownarrow & & \uparrow \psi \\ \pi_{12}(\boldsymbol{S}_{12}) & \xrightarrow{\partial_\xi} & \pi_{11}(\boldsymbol{G}_2). \end{array}$$

Now $\boldsymbol{G}_2$ is embedded in $\boldsymbol{SO}(7)$ and its action on $\boldsymbol{W}_{11}$ extends to that of $\boldsymbol{SO}(7)$; in other words, $\boldsymbol{G}_2$, as a subgroup of $\boldsymbol{SO}(7)$, acts transitively on $\boldsymbol{W}_{11} = \boldsymbol{SO}(7)/\boldsymbol{SO}(5)$ and $\boldsymbol{G}_2 \cap \boldsymbol{SO}(5) = \boldsymbol{Sp}(1)$. This means that if we

[5] It will be shown later that $k = 1$.

extend the structural group of η to $\boldsymbol{SO}(7)$, we get the associated bundle to the extension ξ' of ξ. Therefore we have by 26.4:

$$\partial_\eta(\iota_{12}) = \pm 2^{-a(3)-1} p^*{}_3(\xi')\gamma_3 \bmod 2\text{-torsion}$$

and (6) gives then: $\psi(\alpha) = \pm p^*{}_3(\xi')\cdot\gamma_3$, up to a power of two. Since (25.9) the number $p^*{}_3(\xi')$ is divisible by 15, and since this argument is valid for any $\alpha \in \pi_{11}(\boldsymbol{G}_2)$, the mod C_2 exactness of (d) is established.

26.8. PROPOSITION. *If we have*

$$(7) \qquad \pi_9(\boldsymbol{SO}(7)) \equiv 0, \qquad \pi_{10}(\boldsymbol{U}(5)) \equiv \boldsymbol{Z}_{15} \bmod C_2,$$

then the following congruences mod C_2 *are valid:* $\pi_{10}(\boldsymbol{G}_2) \equiv 0$, $\pi_{10}(\boldsymbol{SO}(5)) \equiv \boldsymbol{Z}_{15}$, $\pi_{10}(\boldsymbol{SO}(n)) \equiv 0$ $(n \geqq 7, n \neq 8)$, $\pi_{10}(\boldsymbol{Sp}(n)) \equiv 0$ $(n \geqq 3)$, $\pi_{10}(\boldsymbol{U}(n)) \equiv 0$ $(n \geqq 6)$.

In this proof, all congruences are mod C_2. We use the following results:

$$(8) \qquad \pi_{n+1}(\boldsymbol{S}_n) \equiv \pi_{n+2}(\boldsymbol{S}_n) \equiv 0 \quad (n \geqq 3), \qquad \pi_{n+3}(\boldsymbol{S}_n) \equiv \boldsymbol{Z}_3 \quad (n \geqq 5),$$

$$(9) \qquad \pi_{10}(\boldsymbol{S}_3) \equiv \boldsymbol{Z}_{15}, \qquad \pi_9(\boldsymbol{S}_3) \equiv \boldsymbol{Z}_3.$$

(For the last equality of (8), see J-P. Serre, Comm. Math. Helv. 27 (1953), 198-232, for the other ones, see [25]; as to (7), see 26.9 and 26.10 below.)

The first equality in (9) shows that in 26.7d, we have $k = 1$ and $\pi_{10}(\boldsymbol{G}_2) \equiv 0$. The fibering $\boldsymbol{G}_2/\boldsymbol{Sp}(1) = \boldsymbol{W}_{11}$, discussed in [4, § 17], together with (9) and the congruence $\pi_i(\boldsymbol{W}_{11}) \equiv \pi_i(\boldsymbol{S}_{11})$ shows that $\pi_9(\boldsymbol{G}_2) \equiv \boldsymbol{Z}_3$. Applying this and (7) to the exact homotopy sequence of the fibering $\boldsymbol{Spin}(7)/\boldsymbol{G}_2 = \boldsymbol{S}_7$ (see [1]), we get the mod C_2 exact sequence

$$0 \to \pi_{10}(\boldsymbol{SO}(7)) \to \boldsymbol{Z}_3 \to \boldsymbol{Z}_3 \to 0;$$

hence $\pi_{10}(\boldsymbol{SO}(7)) \equiv 0$. The mod C_2 exact sequence 26.7c yields then, for $q = 3$, that $\pi_{10}(\boldsymbol{SO}(5)) \equiv \boldsymbol{Z}_{15}$. Since, as is well known, the universal covering $\boldsymbol{Spin}(5)$ of $\boldsymbol{SO}(5)$ is isomorphic to $\boldsymbol{Sp}(2)$, we deduce from 26.7b that $\pi_{10}(\boldsymbol{Sp}(3)) \equiv 0$, and hence also $\pi_{10}(\boldsymbol{Sp}(n)) \equiv 0$ for $n \geqq 3$.

Since $\boldsymbol{SO}(9)/\boldsymbol{SO}(7) = \boldsymbol{W}_{15}$ has, mod C_2, the homotopy groups of $\boldsymbol{S}_{15}$, we have $\pi_{10}(\boldsymbol{SO}(9)) \equiv \pi_{10}(\boldsymbol{SO}(7)) \equiv 0$ and then $\pi_{10}(\boldsymbol{SO}(n)) \equiv 0$ $(n \geqq 9)$ follows from (8), the finiteness of $\pi_{10}(\boldsymbol{SO}(11))$ (see [25]) and the homotopy sequence of $\boldsymbol{SO}(n)/\boldsymbol{SO}(n-1) = \boldsymbol{S}_{n-1}$.

Finally, (7) and 26.7a give $\pi_{10}(\boldsymbol{U}(6)) \equiv 0$, and therefore $\pi_{10}(\boldsymbol{U}(n)) \equiv 0$ for $n \geqq 6$.

26.9. The preceding results (found in Spring 1957) contradict several

of those of [30]. Since then, Toda has made new computations whose outcome (yet unpublished) agrees with the above. They have also been confirmed by Bott (Proc. Nat. Ac. Sci. USA 43 (1957), pp. 933-935) who in particular determines all stable homotopy groups of the classical groups.

26.10. Bott has also shown (to be published) that the sequence 26.7(a) is exact also for the 2-primary components (since $\pi_{2q-2}(\boldsymbol{U}(q))=0$, by Bott, loc. cit., 26.9, this gives $\pi_{2q-2}(\boldsymbol{U}(q-1))=\boldsymbol{Z}_{(q-1)!}$). This implies (see 26.3 and the proof of 26.7) the following generalization of 25.8, 25.9:

THEOREM (Bott). *Let ξ be a $\boldsymbol{U}(k)$-bundle over S_{2q}. Then $c^*_q(\xi)$ is divisible by $(q-1)!$. Let η be a $\boldsymbol{SO}(k)$- (respectively $\boldsymbol{Sp}(k)$-), bundle over over S_{4q}. Then $p^*_q(\eta)$ (respectively $e^*_q(\eta)$), is divisible by $(2q-1)!$.*

Using this theorem, Kervaire has proved more generally that $p^*_q(\eta)$ (respectively $e^*_q(\eta)$) is divisible by $(2q-1)!\cdot 2$ if q is odd (respectively even) (Amer. Jour. Math., vol. 80 (1958), pp. 632-638).

The stable homotopy groups $\pi_{2q-1}(\boldsymbol{U}(k))$, $(k\geqq q)$, and $\pi_{4q-1}(\boldsymbol{SO}(k))$ $(k\geqq 4q+1)$ are infinite cyclic according to Bott (loc. cit. in 26.9). The generator of the first group has the Chern number $c^*_q=\pm(q-1)!$, the generator of the second group has Pontrjagin number p^*_q equal to $\pm(2q-1)!$ if q is even and $\pm(2q-1)!\cdot 2$ if q is odd; similarly for the symplectic groups (with odd and even interchanged). This follows from the preceding theorem, the result of Kervaire and 26.5. The "spinor-method" in 26.5 gives an explicit construction for these generators.

The above theorem would follow by the same argument as 25.8, 25.9 if one could prove that the virtual Todd genus with respect to an *integral* class is an integer. In this respect, compare the conjectures in 25.6. The last one would contain Kervaire's result for the orthogonal groups.

Finally, we remark that the proof of 25.8, 25.9 also applies if the sphere is replaced by a compact connected oriented manifold X whose real Pontrjagin classes p_j vanish for $4j\neq 0$, $\dim X$, and if ξ (respectively η) is a $U(k)$-bundle (respectively $\boldsymbol{O}(k)$- or $\boldsymbol{Sp}(k)$-bundle) whose real Chern (respectively Pontrjagin or symplectic Pontrjagin) classes vanish in all positive dimensions less than $\dim X$.

26.11. Milnor and Kervaire, independently, have deduced from the result of Bott quoted in 26.10 that S_n, endowed with its usual differentiable structure, is not parallelizable if $n\neq 1,3,7$. An easy argument similar to 26.3 shows that if S_{2n-1} is parallelizable, that is if the fibering $\boldsymbol{SO}(2n)/\boldsymbol{SO}(2n-1)$ $=S_{2n-1}$ has a cross section, then there exists a $\boldsymbol{SO}(2n)$-bundle η over S_{2n}

whose Euler-Poincaré class W_{2n} is equal to $1 \cdot \iota_{2n}$. Therefore the theorem of Milnor-Kervaire is a consequence of the

THEOREM (Milnor). *Let ξ be a $\boldsymbol{SO}(2q)$-bundle over S_{2q}, where $q \neq 1, 2, 4$. Then $W_{2q}(\xi)$ is divisible by 2.*

We want to give here a proof for this, different from Milnor's but also using 26.10.

Let η be a $\boldsymbol{SO}(2q)$-bundle whose second Stiefel-Whitney class w_2 vanishes. Then (see beginning of the proof of 26.5), η has a λ-restriction η', where λ is the projection of $\boldsymbol{Spin}(2q)$ onto $\boldsymbol{SO}(2q)$. We denote again by θ the extension of η' by means of the half-spinor representation β. It is a $\boldsymbol{U}(2^{q-1})$-bundle.

LEMMA. *In the previous notations, we have*

(i) $c_q(\theta)/(q-1)! = W_{2q}(\eta)/2$ *if q is odd,*

(ii) $$c_{2k}(\theta)/(2k-1)! = -(tg^{(2k-1)}(0)/((2k-1)!4))p_k(\eta) - W_{4k}(\eta)/2 + R_{2k}(\eta)$$ *if* $q = 2k \geqq 2$,

where $R_{2k}(\eta)$ is a polynomial with rational coefficients in $p_1(\eta), \cdots p_{k-1}(\eta)$, and $tg^{(2k-1)}(0)$ denotes the $(2k-1)$-th derivative of $tg\, x$ at $x=0$.

We keep the notations of 26.5. Since ρ^* is injective, we allow ourselves to omit the symbol ρ^*. The computations of 26.5 show first that

$$\sum \omega_j{}^q = q! \cdot x_1 \cdots x_q/2 + q!\, r'_q(x_1{}^2, \cdots, x_q{}^2), \tag{10}$$

where r'_q has rational coefficients; this can be written

$$\sum \omega_j{}^q = q!\, W_{2q}(\eta)/2 + q! \cdot r_q, \tag{11}$$

r_q being a polynomial in the $p_i(\eta)$ with rational coefficients. On the other hand, $c_q(\theta)$ is the q-th elementary symmetric function in the ω_j's, hence

$$\sum \omega_j{}^q = (-1)^{q-1} q \cdot c_q(\theta) + q!\, s_q,$$

where s_q is a polynomial in the $c_i(\theta)$ $(i < q)$ with rational coefficients. Now, θ is extension of a $\boldsymbol{Spin}(2q)$-bundle η', hence its characteristic ring is contained in the characteristic ring of η'. Since we consider real cohomology, $\rho^*(\lambda): H^*(B_{\boldsymbol{SO}(2q)}) \to H^*(B_{\boldsymbol{Spin}(2q)})$ is an isomorphism, and therefore the $c_i(\theta)$ belong to the characteristic ring of η, which is generated by the $p_i(\eta)$ $(i < q)$ and by $W_{2q}(\eta)$. Thus we get

$$(-1)^{q-1}c_q(\theta)/(q-1)\,! = W_{2q}(\eta)/2 + t_q,$$

where $t_q = r_q - s_q$ is a polynomial in the $p_i(\eta)$ $(2i < q)$ with rational coefficients. It is necessarily zero if q is odd, and this proves (i). In order to prove (ii), we have to compute the coefficient d_k of $p_k(\eta)$ in t_{2k}. By the above, d_k is equal to the coefficient of $p_k(\eta)$ in r_{2k}. Let S be defined by $S(x_1) = -x_1$ and $S(x_i) = x_i$ $(i \geqq 2)$ and put $\sigma_j = S(\omega_j)$. The set $\{\sigma_j\}$ is then the set of forms

$$(\epsilon_1 x_1 + \cdots + \epsilon_{2k} x_{2k})/2, \quad \prod \epsilon_i = -1,$$

(which are the weights of the second half spinor representation), and (10) yields

(12) $$\sum \sigma_j^{2k} = -(2k)\,! \, x_1 \cdots x_{2k}/2 + (2k)\,! \, r'_{2k}(x_1^2, \cdots, x_{2k}^2),$$

hence

(13) $$\sum (\omega_j^{2k} + \sigma_j^{2k}) = (2k)\,! \, 2 \cdot r_{2k},$$

so that $2d_k$ is the coefficient of p_k in

$$((2k)\,!)^{-1} \sum (\omega_j^{2k} + \sigma_j^{2k}).$$

We have clearly

$$\sum_{\epsilon_i = \pm 1} \exp(\epsilon_1 x_1 + \cdots + \epsilon_{2k} x_{2k})/2 = 2^{2k} \prod_{j=1}^{j=2k} \cosh(x_j/2).$$

Let $\{D_j(p_1, \cdots, p_j)\}$ be the multiplicative sequence with $\cosh z^{\frac{1}{2}}/2$ as characteristic power series (this is well defined since $\cosh x$ is an even function in x). Then $2^{-2k+1} d_k$ is for $k \geqq 1$ the coefficient of p_k in D_k. The formula 1.4(10) of [19] yields therefore, (with $2d_0 = 1$),

$$2 \sum_{j \geqq 0} d_j(-z/4)^j = \cosh(\tfrac{1}{2} z^{\frac{1}{2}}) \cdot d(z/\cosh(\tfrac{1}{2} z^{\frac{1}{2}}))/dz,$$

$$2 \sum_{j \geqq 0} d_j(-z/4)^j = 1 - (z^{\frac{1}{2}}/4) \operatorname{tgh}(\tfrac{1}{2} z^{\frac{1}{2}}).$$

Putting $z = -4x^2$, we get then

$$\sum_{j \geqq 1} d_j x^{2j-1} = (1/4) \operatorname{tg} x,$$

and this ends the proof of (ii).

Proof of the theorem. Since the base space of ξ is $\boldsymbol{S}_{2q}$ $(q \neq 1)$, we have $w_2(\xi) = 0$, hence we may apply the lemma to ξ. For q odd, the theorem follows then from (i) and from the divisibility theorem of Bott (26.10). Let now $q = 2k$ be even. We have $tg'(0) = 1$, $tg^{(3)}(0) = 2$, and it is well known, and easily checked, that $tg^{(2k-1)}(0)$ is an integer divisible by 4 for

$k \geqq 3$. Moreover, in (ii) of the lemma we have $R_{2k} = 0$ since $B_\xi = S_{4k}$. Thus, for $q = 2k$, the theorem follows from the lemma and 26.10.

27. Multiplicative properties of the index and consequences.

27.1. *Notation.* Throughout this and the following paragraph, all cohomology groups will be taken with real coefficients and all characteristic classes which occur will be regarded as real classes unless otherwise mentioned.

$\mathfrak{D}$ denotes the class of differentiable bundles ξ, where E_ξ, B_ξ, F_ξ are compact connected oriented differentiable manifolds, the orientation of E_ξ being induced by those of B_ξ, F_ξ taken in that order, and where the fundamental group of B_ξ operates trivially on $H^*(F_\xi)$.

We recall (21.5) that if $\{K_j\}$ is a multiplicative sequence with real coefficients which is strictly multiplicative in ξ, then

$$K(E_\xi) = K(B_\xi) \cdot K(F_\xi) \qquad (\xi \in \mathfrak{D}). \tag{1}$$

We wish to prove a theorem which is a sort of converse to 21.5.

27.2. THEOREM. *Let F be a compact connected oriented differentiable manifold. If $\{K_j\}$ is a multiplicative sequence of polynomials with real coefficients, for which (1) holds in every bundle $\xi \in \mathfrak{D}$ such that $F_\xi = F$, then $\{K_j\}$ is strictly multiplicative in each of these bundles.*

The proof uses essentially the theorem of Thom [29, Corollaire II 30] that every real cohomology class of B_ξ is a finite linear combination of real cohomology classes representable by submanifolds. By definition, a real cohomology class is representable by a submanifold if and only if it corresponds by Poincaré duality to a real homology class containing the fundamental cycle of a compact oriented differentiable manifold differentiably imbedded in B_ξ.

The strictly multiplicative behavior of $\{K_j\}$ is obviously true if $\dim B_\xi = 0$. Let us make the induction hypothesis that it is proved for $\dim B_\xi < n$. Then we will prove it, using (1), for an n-dimensional manifold B_ξ. Let $\hat{p}_i$ be the Pontrjagin classes of the bundle along the fibres of ξ. We have to show that

$$\Big(\sum_{j=0}^{\infty} K_j(\hat{p}_1, \cdots, \hat{p}_j)\Big)^{\natural} = K(F_\xi) \cdot 1,$$

where $\natural : H^*(E_\xi) \to H^*(B_\xi)$ is the integration over the fibre (8.1). We can restrict the bundle ξ to every submanifold Y of B_ξ. Since integration over the fibre and restriction commute (8.3), we obtain by our induction hypothesis

that for $\dim Y < \dim B_\xi$ the restriction $(\sum_{j=0}^{\infty} K_j(\hat{p}_1, \cdots, \hat{p}_j))^\natural$ to Y equals $K(F_\xi) \cdot 1$, where 1 denotes now the unit of the cohomology ring of Y. This, together with the above mentioned theorem of Thom, implies

$$(\sum_{j=0}^{\infty} K_j(\hat{p}_1, \cdots, \hat{p}_j))^\natural = K(F_\xi) \cdot 1 + c,$$

where $c \in H^n(B_\xi)$.

We denote the Pontrjagin classes of B_ξ by p'_i and those of E_ξ by p_i. Then (compare the proof of 21.5)

$$\begin{aligned} K(E_\xi) &= (\sum_{j=0}^{\infty} K_j(p_1, \cdots, p_j))[E_\xi] \\ &= ((K(F_\xi) \cdot 1 + c) \cdot \sum_{t=0}^{\infty} K_t(p'_1, \cdots, p'_t))[B_\xi] \\ &= K(F_\xi)(\sum_{t=0}^{\infty} K_t(p'_1, \cdots, p'_t))[B_\xi] + c[B_\xi] \\ &= K(F_\xi) \cdot K(B_\xi) + c[B_\xi]. \end{aligned}$$

According to (1) we have $K(E_\xi) = K(B_\xi) K(F_\xi)$ and thus obtain $c = 0$, which completes the proof.

It was proved recently [12] for the index τ and a bundle $\xi \in \mathfrak{D}$ that

$$\tau(E_\xi) = \tau(B_\xi) \cdot \tau(F_\xi). \tag{2}$$

Since the index τ equals the genus L defined by the multiplicative sequence $\{L_j\}$, see [19, § 8], we get in virtue of the preceding theorem the following result:

27.3. THEOREM. *The sequence $\{L_j\}$ is strictly multiplicative in every $\xi \in \mathfrak{D}$.*

If $\xi \in \mathfrak{D}$ and $\natural$ is the integration over the fibre in ξ, we have therefore

$$(L_j(\hat{p}_1, \cdots, \hat{p}_j))^\natural = 0 \qquad (4j \neq \dim F_\xi),$$

$$(L_j(\hat{p}_1, \cdots, \hat{p}_j))^\natural = \tau(F_\xi) \cdot 1 \qquad (4j = \dim F_\xi).$$

27.4. *Examples.*

1) $\dim F_\xi = 2$. In this case, $\tau(F_\xi)$ vanishes. $L_j(\hat{p}_1, 0, \cdots, 0)$ is a non-zero multiple of $\hat{p}_1{}^j$, see [19, § 1]. Since $\hat{p}_1 = \hat{W}_2{}^2$, where $\hat{W}_2$ is the Euler class of the bundle along the fibres, we get (in real cohomology)

$$(\hat{W}_2{}^{2j})^\natural = 0, \qquad j = 1, 2, 3, \cdots.$$

If B_ξ is of dimension $4k-2$ and hence E_ξ of dimension $4k$, then $\hat{W}_2^{2k}=0$.

2) $\dim F_\xi = 3$. We get for the Pontrjagin class $\hat{p}_1$ of the bundle along the fibres that $(\hat{p}_1{}^j)^\natural$ $(j=1,2,3,\cdots)$, vanishes in real cohomology.

3) $\dim F_\xi = 4$. We have $45\cdot L_2(\hat{p}_1,\hat{p}_2)=7\hat{p}_2-\hat{p}_1{}^2$. Thus

$$(7\hat{p}_2-\hat{p}_1{}^2)^\natural=0.$$

This equation is proved in real cohomology and not known in integral cohomology. Again $\hat{p}_2=\hat{W}_4{}^2$, where $\hat{W}_4$ is the Euler class of the bundle along the fibres. If $\dim B_\xi=4$ (and hence $\dim E_\xi=8$), then

$$7\cdot\hat{W}_4{}^2=\hat{p}_1{}^2.$$

27.5. *Remark.* The strict multiplicativity of $\{L_j\}$ was proved in 22.9 for bundles $\xi\in\mathfrak{D}$ with $F_\xi=G/U$ $(\operatorname{rank} G=\operatorname{rank} U)$ and G as structural group. It was also shown (§ 23) in this special case that the sequence $\{A_j\}$ is strictly multiplicative provided $A(F_\xi)=0$. It might be conjectured that in every fibre bundle $\xi\in\mathfrak{D}$ the vanishing of $A(F_\xi)$ implies that of $A(E_\xi)$. As a consequence, we would have (27.2) that $\{A_j\}$ is strictly multiplicative in every fibre bundle $\xi\in\mathfrak{D}$ with $A(F_\xi)=0$.

28. A uniqueness theorem on the index.

28.1. In this paragraph, we shall prove that the sequence $\{L_j\}$ is essentially the only multiplicative sequence with coefficients in a field of characteristic 0 giving rise to a genus which is multiplicative in fibre bundles. Throughout this paragraph, we keep the notations of 27.1.

Let M be a $4k$-dimensional compact oriented differentiable manifold and p_i its Pontrjagin classes, which may be written formally as elementary symmetric functions:

$$1+p_1+\cdots+p_k=(1+\beta_1)\cdots(1+\beta_k).$$

Then the number $s(M)$ is defined by

$$s(M)=(\beta_1{}^k+\cdots+\beta_k{}^k)[M], \qquad \text{(see [19, § 6.3])}.$$

28.2. LEMMA. *Let $\xi\in\mathfrak{D}$ be a principal $\boldsymbol{U}(q)$-bundle over a 4-dimensional base space and c_i its Chern classes. If $2q+2=4k\geqq 8$, then*

$$s(E_\xi/\boldsymbol{U}(1)\times\boldsymbol{U}(q-1))=(-q(q+2)c_2+(\binom{q+1}{2}-1)c_1{}^2)[B_\xi].$$

For the proof, we use the notations of 15.1. We always take into

account that B_ξ is 4-dimensional, and hence, for example, $c_i=0$ for $i>2$ and

(1) $$\pi^*(1+c_1+c_2)=(1+x_1)(1+x_2)\cdots(1+x_q).$$

Furthermore

(2) $$x_1^q-x_1^{q-1}\pi^*(c_1)+x_1^{q-2}\pi^*(c_2)=0.$$

Letting $a=\sum_{j=1}^{q}(x_j-x_1)^{q+1}$, we get by (1) that

(3) $$(-1)^q a=-q\cdot x_1^{q+1}+(q+1)x_1^q\cdot\pi^*(c_1) -\binom{q+1}{2}x_1^{q-1}\cdot\pi^*(c_1^2-2c_2).$$

We infer readily from (2) and (3) that

$$(-1)^q a=(q+2)q\cdot x_1^{q-1}\pi^*(c_2)+(1-\binom{q+1}{2})\cdot x_1^{q-1}\cdot\pi^*(c_1^2).$$

We may put $a=\rho^*(b)$ and $x_1=\rho^*(\gamma_1)$. Then the preceding formula yields

(4) $$b=-q(q+2)(-\gamma_1)^{q-1}\sigma^*(c_2)+(\binom{q+1}{2}-1)\cdot(-\gamma_1)^{q-1}\sigma^*(c_1^2)$$

M will denote the total space of the bundle $\theta=(E_\xi/\boldsymbol{U}(1)\times\boldsymbol{U}(q-1), B_\xi,\boldsymbol{P}_{q-1}(\boldsymbol{C}))$. The tangent bundle of M is the Whitney sum of $\hat{\theta}$, the bundle along the fibres of θ, and of the tangent vector bundle of B_ξ lifted by σ. We have for the total Pontrjagin classes

(5) $$p(M)=p(\hat{\theta})\cdot\sigma^*p(B_\xi)=p(\hat{\theta})\cdot(1+\sigma^*p_1(B_\xi)).$$

In 15.1, one finds a formula for $\rho^*c(\eta')$ which yields

(6) $$\rho^*p(\hat{\theta})=\prod_{j=1}^{q}(1+(x_j-x_1)^2).$$

By (5) and (6), we get for $2q+2=4k$

$$s(M)=(b+(\sigma^*(p_1(B_\xi))^k))[M].$$

Since $(p_1(B_\xi))^k=0$ for $k\geqq 2$, we have $s(M)=b[M]$ which, by (4), completes the proof because the value of $(-\gamma_1)^{q-1}$ on the oriented fibres of θ equals 1.

28.3. We are going to construct a special base sequence for the algebra $\Omega\otimes\boldsymbol{Q}$ of Thom [29]. Consider over $X=\boldsymbol{P}_2(\boldsymbol{C})$ the differentiable principal $\boldsymbol{U}(q)$-bundle $\xi(q)$ which is the Whitney sum of $q-2$ trivial $\boldsymbol{U}(1)$-bundles and of the two principal $\boldsymbol{U}(1)$-bundles with g and $-g$ respectively as first

Chern classes, where g is the cohomology class dual to a complex projective line imbedded in X. The Chern classes of $\xi(q)$ are

$$c_1 = 0, \qquad c_2 = -g^2.$$

For $2q+2=4k \geqq 8$, let E^{4k} be the $4k$-dimensional manifold fibred with $\boldsymbol{P}_{q-1}(\boldsymbol{C})$ as fibre and associated to $\xi(q)$; i.e.

$$E^{4k} = E_{\xi(q)}/\boldsymbol{U}(1) \times \boldsymbol{U}(q-1), \qquad (2q+2=4k \geqq 8).$$

According to the preceding lemma, we have

$$(7) \qquad s(E^{4k}) = (2k+1)(2k-1) = 4k^2 - 1 \neq 0.$$

For $k=1$, we put $E^4 = \boldsymbol{P}_2(\boldsymbol{C})$ and then (7) holds also in this case.

By [19, § 6], the sequence $\{E^{4k}\}$, $(k=1,2,3,\cdots)$, of $4k$-dimensional manifolds is a base sequence of the algebra $\Omega \otimes \boldsymbol{Q}$ of Thom; and we have in terms of the usual base sequence $\boldsymbol{P}_{2k}(\boldsymbol{C})$

$$(8) \qquad E^{4k} = (2k-1)\boldsymbol{P}_{2k}(\boldsymbol{C}) + \text{composite terms in the } \boldsymbol{P}_{2j}(\boldsymbol{C}), j<k.$$

28.4. THEOREM. *Let $\{K_r(p_1,\cdots,p_r)\}$ be a multiplicative sequence of polynomials with coefficients in a field of characteristic* 0 *and suppose that the corresponding genus K satisfies the equation $K(E)=K(B)\cdot K(F)$ for all differentiable fibre bundles in $\mathfrak{D}$, or equivalently* (27.2), *that $\{K_r(p_1,\cdots,p_r)\}$ is strictly multiplicative in $\mathfrak{D}$. Put $a = K(\boldsymbol{P}_2(\boldsymbol{C}))$. Then*

$$(9) \qquad K(Y) = a^r \cdot L(Y), \qquad (4r = \dim Y)$$

for all $4r$-dimensional compact oriented differentiable manifolds Y, and moreover

$$(10) \qquad K_r(p_1,\cdots,p_r) = a^r L_r(p_1,\cdots,p_r), \qquad (r=1,2,3\cdots).$$

We prove (9) by induction over r. It is true for $r=1$ since $\boldsymbol{P}_2(\boldsymbol{C})$ generates Ω^4. Suppose it is proved for all Y with $\dim Y < 4r$. The vector space $\Omega^{4r} \otimes \boldsymbol{Q}$ over the rationals is generated by E^{4r} and "composite" manifolds M^{4r} which are cartesian products of lower dimensional manifolds. Since K and L are both multiplicative in cartesian products, (9) is true on the composite manifolds M^{4r} by induction hypothesis, and since K and L are also both multiplicative in differentiable fibre bundles, (9) is true on E^{4r} too (again we have used the induction hypothesis). Thus (9) holds on $\Omega^{4r} \otimes \boldsymbol{Q}$.

This proves (9) in full generality which implies (10), [19, Satz 6.5.1].

Appendix I.

29. The different definitions of the Chern classes.

29.1. *Orientation conventions.* In the n-dimensional complex vector space V with coordinates $z_j = x_j + iy_j$, we take as usual the orientation defined by the order $x_1, y_1, \cdots, x_n, y_n$. This determines also an orientation for complex analytic manifolds as well as for the $2n-1$ dimensional sphere S_{2n-1} of unit vectors with respect to some hermitian metric on V. The image of the fundamental cycle of S_{2n-1} thus defined in $\pi_{2n-1}(S_{2n-1})$, or $H_{2n-1}(S_{2n-1}, \boldsymbol{Z})$, or $H_{2n-1}(V-0, \boldsymbol{Z})$, (0 being the origin in V), and the element of $H^{2n-1}(S_{2n-1}, \boldsymbol{Z})$ or $H^{2n-1}(V-0, \boldsymbol{Z})$ taking the value 1 on it will be called the canonical generator of the corresponding group.

Let $W_{n,n-q+1}$ be the complex Stiefel manifold of ordered systems of $n-q+1$ orthonormal vectors in $\boldsymbol{C}^n$ $(1 \leqq q \leqq n)$. We know that its first non-vanishing homotopy group is in dimension $2q-1$ and is infinite cyclic. Now $W_{n,n-q+1}$ is fibered by $W_{q,1} = S_{2q-1}$, with base $W_{n,n-q}$; the projection assigning to each $(n-q+1)$-frame the $(n-q)$-frame formed by its first $n-q$ elements. The fibre may thus be identified with the set of unit vectors in $\boldsymbol{C}^q$ and its injection in $W_{n,n-q+1}$ induces isomorphisms for the $(2q-1)$-st homotopy or homology or cohomology groups. The element corresponding to the canonical generator previously defined will also be called the canonical generator.

Similarly, let $W^*_{n,n-q+1}$ be the manifold of ordered systems of $n-q+1$ linearly independent vectors in $\boldsymbol{C}^n$; it has $W_{n,n-q+1}$ as a deformation retract; let $e_1, \cdots, e_{n-q}$ be independent vectors, and let V be a q-dimensional subspace supplementary to the space spanned by the e_i's. The subspace U of $W^*_{n,n-q+1}$ made of the systems (f_j), $(j=1, \cdots, n-q+1)$, for which $f_j = e_j$ $(j \leqq n-q)$ and f_{n-q+1} is in V may be identified with $V-0$. Its injection in the Stiefel manifold is an isomorphism for homotopy or homology in dimension $2q-1$ and we define as before the canonical generator of $H_{2q-1}(W^*_{n,n-q+1}, \boldsymbol{Z})$ and $\pi_{2q-1}(W^*_{n,n-q+1})$.

29.2. *The Hopf fibering.* (x_i), $(1 \leqq i \leqq n+1)$, are the coordinates of $\boldsymbol{C}^{n+1}$ and the homogeneous coordinates in the complex projective space $\boldsymbol{P}_n$. By the Hopf fibering over $\boldsymbol{P}_n$, we mean here $\boldsymbol{C}^{n+1}-0$ endowed with the usual $\boldsymbol{C}^* = \boldsymbol{GL}(n,1)$ bundle structure. e will denote a hyperplane with the positive orientation or the corresponding homology class and $e^* \in H^2(\boldsymbol{P}_n, \boldsymbol{Z})$ will be the dual cohomology class. Let U_i be the set of points in $\boldsymbol{P}_n$ for

which $x_i \neq 0$, $(1 \leqq i \leqq n+1)$; using the usual conventions for the transition functions of a bundle [19, § 3.2.a] and of a line bundle associated to a divisor D [19, § 15.2] we see that in $U_i \cap U_j$ the Hopf fibering is given by $f_{ij} = x_i/x_j$, whereas the bundle $\{e\}$ associated to e is given by $g_{ij} = x_j/x_i$; thus the Hopf fibering is the inverse of $\{e\}$. We recall that the Hopf fibering over $\boldsymbol{P}_n$ is a $2n$-universal bundle for $\boldsymbol{C}^*$ or $\boldsymbol{U}(1)$.

29.3. *The definitions of Chern classes.* Including the definition (9.1), there are apparently seven definitions of Chern classes, which we proceed to list now; ${}^ic_j$ will be the j-th Chern class according to the i-th definition, and ${}^ic$ the sum of the ${}^ic_j$.

(1) *The definition* (9.1) *of this paper.* It may also be formulated in the following way: in the Hopf fibering, we put ${}^1c_1 = -\tau(x)$, where x is the canonical generator of $H^1(\boldsymbol{C}^*, \boldsymbol{Z})$; for a general $\boldsymbol{C}^*$-bundle, we use the characteristic map; for a general $\boldsymbol{GL}(n, C)$ bundle, we go over to the bundle of flags and take the elementary symmetric functions in the Chern classes of the different line bundles in which the lifted bundle decomposes.

(2) *The definition of* [19]: it is quite similar to (1), except that we put ${}^2c_1 = -e^*$ in the Hopf fibering.

(3) *The obstruction definition.* Given a complex vector bundle $(E, B, \boldsymbol{C}^n)$, we consider the associated bundle $(E', B, \boldsymbol{W}_{n,n-q+1})$. The first obstruction to the construction of a cross section (B is supposed to be a complex here) is an element of $H^{2q}(B, \pi_{2q-1}(\boldsymbol{W}_{n,n-q+1}))$. We identify the coefficient group with $\boldsymbol{Z}$ by sending the canonical generator onto 1, and thus get a class ${}^4c_q \in H^{2q}(B, \boldsymbol{Z})$.

This convention was introduced in [20], and was recalled at the beginning of [11] but was not made in [9], where consequently the obstruction classes are defined up to sign only.

(4) *The definition of* [6]: in the Hopf fibering, considered as universal bundle, we put ${}^4c_1 = \tau(x)$, and then proceed as in (1).

(5) *Schubert systems.* Let $\boldsymbol{H}(n, N)$ be the complex Grassmann manifold of n-dimensional subspaces of $\boldsymbol{C}^{n+N}$. It is the base space of the $2N$-universal bundle $(\boldsymbol{U}(n+N)/\boldsymbol{U}(N), \boldsymbol{H}(n, N), \boldsymbol{U}(n))$ for $\boldsymbol{U}(n)$, where $\boldsymbol{U}(N)$, (respectively $\boldsymbol{U}(n)$), is the subgroup of $\boldsymbol{U}(n+N)$, leaving the n first (respectively N last) coordinate vectors fixed. For $n=1$, we have the Hopf fibering. We take as universal ${}^5c_q$ the dual class to the Schubert cycle of dimension $2(nN-q)$ represented by the symbol $(N-1, \cdots, N-1, N, \cdots, N)$,

(q times $N-1$, $(n-q)$ times N). By the intersection properties of Schubert varieties, ${}^5c_q$ is also the cohomology class taking the value 1 on the Schubert cycle $(0,\cdots 0,1,\cdots,1)$ ($(n-q)$ times 0, q times 1), and zero on all other Schubert cycles of Ehresmann's cell decomposition of $\boldsymbol{H}(n,N)$; (Schubert cycles are defined for instance in [9, 10, 11].) This defines ${}^5c$ in the universal bundle; for a general bundle, we take its image by the characteristic map. For $n=1$, the Schubert symbol $(N-1)$ represents the hyperplane of $\boldsymbol{P}_N(\boldsymbol{C})=\boldsymbol{H}(1,N)$. Thus, in the Hopf fibering, we have ${}^5c_1=e^*$.

(6) *Definition by means of differential forms.* This leads to real cohomology classes, defined for differentiable bundles. Let ξ be a differentiable principal $\boldsymbol{U}(n)$-bundle and let Ω_{ij} $(1\leqq i,j\leqq n)$ be the curvature forms of a connexion on E_ξ. We then consider the (mixed) differential form:

$$\Psi=\sum\Psi_q=\det|\,\mathrm{Id}+(2\pi i)^{-1}\Omega_{ij}\,|$$

(Ψ_q of degree q; the product in the determinant is of course the exterior product). It defines a form on B_ξ, which is closed. The image of Ψ_q in $H^{2q}(B_\xi,\boldsymbol{R})$ is by definition ${}^6c_q$. This definition is introduced in [9] (our Ψ_q is the Ψ_{n-q+1} of Chern), although in an apparently slightly more restrictive way. Chern considers only bundles of (tangential) orthonormal frames to a hermitian manifold and a special connexion characterized by a property of its torsion tensor [9, p. 111]. However, by a theorem of Weil, whose proof is reproduced in [10, pp. 58-59], the cohomology class of Ψ_q is independent of the particular connexion chosen in ξ.

(7) *Definition by transgression.* ${}^7c_q$ is the image by transgression of the canonical generator of $H^{2q-1}(\boldsymbol{W}_{n,n-q+1},\boldsymbol{Z})$ in the bundle with fiber $\boldsymbol{W}_{n,n-q+1}$ associated to a given complex vector bundle.

The purpose of this Appendix is to prove the

29.4. THEOREM. *Let ${}^ic_j$ be the j-th Chern class of a bundle $(E,B,\boldsymbol{U}(n))$ with respect to the i-th definition $(j=1,\cdots,n;\, i=1,\cdots,7)$. Then* ${}^1c_j={}^2c_j={}^3c_j=(-1)^j\cdot{}^4c_j=(-1)^j\cdot{}^5c_j=(-1)^j\cdot{}^6c_j=-{}^7c_j$.

29.5. *Remarks.* (a) All these definitions have the naturality property: if $f\colon\xi\to\eta$ is a homomorphism, then $\bar{f}^*({}^ic(\eta))={}^ic(\xi)$, where $\bar{f}\colon B_\xi\to B_\eta$ is induced by f. This is obvious for $i=1,2,4,5,7$, and standard for $i=3$; for $i=6$, it follows by the theorem of Weil quoted above, because if Ω_{ij} are the curvature forms of a connexion $\mathcal{C}$ on E_η, then their images on E_ξ under f will be the curvature forms of the connexion induced from $\mathcal{C}$ by f.

(b) Let a, b $(1 \leqq a, b \leqq 7)$ be given, and assume that ${}^a c_1 = \epsilon \cdot {}^b c_1$ $(\epsilon = \pm 1)$ and that ${}^a c$ obeys duality.[6] Then ${}^b c$ obeys duality if and only if ${}^a c_j = \epsilon^j \cdot {}^b c_j$, $(1 \leqq j \leqq n)$. This is readily seen by using the bundle of flags. Thus ${}^i c$ has the duality property for $i \leqq 6$, but not for $i = 7$. For $i = 1, 2, 4$, the duality property follows immediately from an identity between elementary symmetric functions (see e.g. [6]). In the course of the proof of the theorem we shall *use* the fact that ${}^5 c$ obeys duality, which is proved in [11], and therefore we do not provide a new proof for it. We note in passing that in the introduction of [11], Chern classes are defined as obstructions (with signs) but the proof for duality is carried out for Schubert cocycles. However, our 29.4 shows that the obstruction classes obey duality, a fact for which there is, to our knowledge, no direct proof in the literature.

29.6. LEMMA. *In the Hopf fibering, we have* ${}^7 c_1 = + e^* = - {}^3 c_1$.

In view of a general fact about transgression and obstructions, recalled below (29.7), it is in fact enough to prove one equality, but both may be easily checked directly: As to the first one, we put $a = \sum x_i \cdot \bar{x}_i$ and consider

$$\Omega = (i/2\pi)(a^{-1} \cdot \sum dx_i \wedge d\bar{x}_i - a^{-2} \cdot \sum x_i \cdot \bar{x}_j \, dx_j \wedge d\bar{x}_i);$$

it is the imaginary part, multiplied by $1/\pi$, of the Fubini metric

$$ds^2 = a^{-1} \cdot \sum dx_i \cdot d\bar{x}_i - a^{-2} \cdot \sum x_i \bar{x}_j \, dx_j \cdot d\bar{x}_i. \tag{1}$$

Ω is a closed real form of type $(1,1)$ on $\boldsymbol{P}_n$ and we integrate it on the projective line $\boldsymbol{P}_1$: $x_3 = \cdots = x_{n+1} = 0$ with homogeneous coordinates (x_1, x_2); leaving out $(0,1)$, we replace $\boldsymbol{P}_1$ by the cross section $x_1 = 1$ and get for the integral

$$(i/2\pi) \int_{\boldsymbol{P}_1} (1 + x_2 \cdot \bar{x}_2)^{-2} \cdot dx_2 \wedge d\bar{x}_2 = 1.$$

Hence Ω represents e^*. On the other hand, we have

$$\Omega = (i/2\pi) d\bar{\partial} \log a = d((i/2\pi) a^{-1} \cdot \sum x_i \cdot d\bar{x}_i),$$

and the restriction of $(i/2\pi)\bar{\partial} \log a$ to the fibre $x_2 = \cdots = x_{n+1} = 0$ is $i(2\pi\bar{x}_1)^{-1} d\bar{x}_1$, whose integral on the positively oriented unit circle is again 1; this proves the first equality.

${}^3 c_1$ is the first obstruction to the construction of a cross section in the Hopf fibering and we only need to know its value on $\boldsymbol{P}_1$. Over this line, we consider the cross section defined (except at $(0,1)$) by $(x_1, x_2) \to (1, x_2/x_1)$.

---

[6] By duality we mean the multiplication theorem 9.7(6).

Around $(0,1)$, we take as product representation of the bundle the one which identifies 1 on the typical fibre with $(x_1/x_2, 1)$; then we see easily that around $(0,1)$, the map $\boldsymbol{P}_1 \to \boldsymbol{C}^*$ which defines the value of the obstruction cochain on a 2 simplex having $(0,1)$ in its interior, is of the form $z \to 1/z$; this value will therefore be -1, and so will be that of the obstruction cochain on $\boldsymbol{P}_1$, hence the second equality.

29.7. *Proof of the theorem.* By the definitions, the lemma and naturality, we have

$$^1c_1 = {}^2c_2 = {}^3c_1 = -{}^4c_1 = -{}^5c_1 = -{}^7c_1$$

since all these classes are equal to $-e^*$ in the Hopf fibering. For $i = 1, 2, 4, 5$, ^ic satisfies duality, and we get therefore (see 29.5b)

$$(2) \qquad {}^1c_i = {}^2c_i = (-1)^i \cdot {}^4c_i = (-1)^i \cdot {}^5c_i \qquad (1 \leqq i \leqq n).$$

The equality

$$(3) \qquad {}^3c_j = -{}^7c_j, \qquad (1 \leqq j \leqq n),$$

follows from the more general fact that in a fibre bundle, "transgression is minus obstruction" for a proof of which we refer to [26, § 37.16].

The $2N$-universal bundle for $\boldsymbol{U}(n)$ over $\boldsymbol{H}(n,N)$ is $(\boldsymbol{W}_{n+N,n}, \boldsymbol{H}(n,N), \boldsymbol{U}(n))$ which may also be written as $(\boldsymbol{U}(n+N)/\boldsymbol{U}(N), \boldsymbol{H}(n,N), \boldsymbol{U}(n))$, where $\boldsymbol{U}(N)$, (respectively $\boldsymbol{U}(n)$), is the subgroup of $\boldsymbol{U}(n+N)$ leaving the n first (respectively N last) coordinate vectors fixed. Let (u_{ij}), $(1 \leqq i, j \leqq n+N)$, be the standard coordinates in the matrix space. Following Chern [9], we denote by θ_{ab} the left invariant Maurer-Cartan form on $U(n+N)$ which is equal to du_{ba} (and not du_{ab}) at the neutral element. In these notations, we have

$$(4) \qquad d\theta_{ab} = \sum_1^{n+N} \theta_{ai} \wedge \theta_{ib}; \qquad \theta_{ba} = -\bar{\theta}_{ba}.$$

It is easily seen that for $1 \leqq a, b \leqq n$, the forms are right invariant under $\boldsymbol{U}(N)$ and satisfy

$$\theta_{ab} \cdot u = \sum_{ij} u_{ia} \theta_{ij} (u^{-1})_{bj} \qquad (u \in \boldsymbol{U}(n), u = (u_{ij})),$$

hence they induce forms on $\boldsymbol{U}(n+N)/\boldsymbol{U}(N)$ which define there a connexion for the $\boldsymbol{U}(n)$-bundle structure. (4) shows that its curvature forms are

$$\Omega_{ij} = \sum_{k=n+1}^{k=N} \theta_{ik} \wedge \theta_{kj}, \qquad (1 \leqq i, j \leqq n).$$

Thus, by definition

$$\Psi=\sum\Psi_q=\det|\,\mathrm{Id}+(2\pi i)^{-1}\Omega_{ij}\,|$$

represents ${}^6c$ in the universal bundle. By a computation which we shall not reproduce, Chern ([9], Chap. II, § 2, [10], p. 77) has shown that Ψ_q is equal to ${}^5c_q$; hence

$$(5)\qquad {}^5c_j={}^6c_j, \qquad (j=1,\cdots,n),$$

first in the universal bundle, and then by naturality in all differentiable $U(n)$-bundles.

To conclude the proof, we have to compare the obstruction classes with one of the six other types, and it will be enough to show that

$$(6)\qquad {}^3c_j=(-1)^j\cdot{}^6c_j, \qquad (1\leqq j\leqq n).$$

We have first

$${}^3c_j=\epsilon_j\cdot{}^6c_j \qquad (\epsilon_j=\pm1, j=1,\cdots,n),$$

with ϵ_j depending only on j; by naturality, we have only to prove this in the universal bundle; there it follows from (2), (3), (5), and from the fact that ${}^7c_j$ and ${}^4c_j$ both generate the kernel of $\rho^*(U(j-1),U(n))$ in dimension $2j$, which is infinite cyclic. The proof of this is identical with that of a similar statement on Stiefel-Whitney classes [3, Lemme 5.1] and will be left to the reader.

To determine ϵ_j, it is then enough to compute the 2 classes ${}^3c$ and ${}^6c$ for one bundle with Chern classes not zero or of order 2. To this end, we take the tangent bundle of P_n. By [9, p. 118], we have

$$\Psi_j=(2\pi i)^{-j}\binom{n+1}{j}\Lambda^j$$

for the torsionless connexion associated to the Fubini metric, where Λ is the exterior form obtained from the metric (1) by replacing the symmetric products by exterior products; in the notation of 29.6, we have, therefore, $\Omega=i(2\pi)^{-1}\cdot\Lambda$ and

$$\Psi_j=\binom{n+1}{j}(-1)^j\Omega^j.$$

We have seen in the proof of 29.6 that Ω represents e^* and, consequently,

$$(7)\qquad {}^6c_j=\binom{n+1}{j}(-1)^j(e^*)^j, \qquad (1\leqq j\leqq n),$$

as also follows from 15.1 and (2), (5). On the other hand, a direct con-

struction of vector fields in [9, Theorem 13] shows that

$$(8)\qquad {}^{s}c_j = \binom{n+1}{j}(e^*)^j, \qquad (j=1,\cdots,n),$$

whence $\epsilon_j = (-1)^j$ and (6). Of course we must check that in the proof of (8), the indices of singularities are counted with the proper sign conventions; this presents no difficulty, but for the sake of completeness, we outline Chern's construction. Let

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ a_{11} & a_{12} & & a_{1,n+1} \\ & \cdot\ \cdot\ \cdot\ \cdot & & \\ a_{n1} & a_{n2} & \cdots & a_{n,n+1} \end{pmatrix}$$

be a matrix with constant coefficients and all minors of degree j $(1 \leqq j \leqq n+1)$ different from zero. Let H_j be the j-dimensional projective subspace defined by

$$\sum_k a_{ik}x_k = 0 \qquad (1 \leqq i \leqq n-j).$$

We denote by $\mathfrak{v}_j$ the vector field on $\boldsymbol{C}^{n+1} - 0$ defined by

$$\mathfrak{v}_j(x) = \sum_k a_{jk}x_k\mathfrak{e}_k, \qquad ((\mathfrak{e}_k) \text{ canonical basis of } \boldsymbol{C}^{n+1}).$$

It is invariant under $x \to a \cdot x$ $(a \in \boldsymbol{C}^*)$ and defines a vector field $\mathfrak{w}_j$ on $\boldsymbol{P}_n$. The vectors $\mathfrak{w}_j$ $(j \leqq r)$ are dependent at a point P with homogeneous coordinates (x_i) if and only if the $r+1$ vectors $\mathfrak{x} = \sum x_i\mathfrak{e}_i$ and $\mathfrak{v}_1, \cdots, \mathfrak{v}_r$ are dependent at $x = (x_i)$, which is equivalent with the vanishing of $n+1-r$ homogeneous coordinates of P.

Let now j be fixed and put $r = n - j + 1$. On H_j, $\mathfrak{w}_1, \cdots, \mathfrak{w}_{r-1}$ are independent everywhere whereas $\mathfrak{w}_1, \cdots, \mathfrak{w}_r$ are dependent at $\binom{n+1}{j}$ points. We use these vector fields to compute the value of ${}^{s}c_j$ on H_j; since we already know that it is equal to $\pm\binom{n+1}{j}$, we have only to show that the singular points have non-negative indices. Let Q be a singular point, and W a neighborhood of Q on H_j. Using the fields $\mathfrak{w}_1, \cdots, \mathfrak{w}_{r-1}$ which are also independent at Q we see immediately that the map $W - Q \to \boldsymbol{W}^*_{n,r}$ leading to the index is homotopic to a complex analytic map of $W - Q$ into a subspace of $\boldsymbol{W}^*_{n,r}$ of the type of the space U introduced in 29.1, and that the resulting map of $W - Q$ in $\boldsymbol{C}^j - 0$ extends analytically to Q and maps it onto the origin; hence it has a positive degree, and the index is $\geqq 0$ according to the conventions of 29.1 and 29.3(3).

29.8. *Remarks.* (a) The obstruction classes ${}^3c_j$ are the obstructions to the construction of *contravariant* vector fields. Hodge [20] uses covariant vector fields and, by 29.4 and 10.6a, or directly, the resulting classes are our ${}^4c_j$.

(b) According to Hodge [20] and Nakano (Mem. Coll. Sci. Kyoto 29 (1955), 145-149), the canonical classes of Eger-Todd are to be identified with the Schubert classes ${}^5c_j$.

(c) The lack of sign conventions for obstruction classes in [9] leads to a slight inaccuracy for the Chern classes of $\boldsymbol{P}_n$; Theorem 13 gives $c_j = \binom{n+1}{j}(e^*)^j$ and Theorem 12 gives the class of Ψ_j; as we have seen, these differ by $(-1)^j$.

29.9. Finally, to be complete, we list some properties or alternate definitions of the first Chern class. For the notations, and concepts used here, see [19].

(a) Let $\boldsymbol{C}^*_c$ be the sheaf of germs of continuous $\boldsymbol{C}^*$-valued functions on the space B. A complex line bundle over B is represented by an element $\xi \in H^1(B, \boldsymbol{C}^*_c)$ and we have

$$ {}^1c_1 = \delta\xi, $$

where δ is the coboundary operator $H^1(B, \boldsymbol{C}^*_c) \to H^2(B, \boldsymbol{Z})$ associated to the exact sequence $0 \to \boldsymbol{Z} \to \boldsymbol{C}_c \xrightarrow{e} \boldsymbol{C}^*_c \to 0$, where e is the exponential map [19, § 4.3.1].

(b) Let V be an oriented m-dimensional manifold, B an oriented $(m-2)$-dimensional submanifold, η the normal bundle oriented in such a way that orientation of B plus orientation of η gives the orientation of V. It has then a complex structure compatible with its orientation, determined up to isomorphism; its class ${}^1c_1$ is dual to the homology class defined by B [19, § 4.8.1].

(c) Kodaira has introduced the following definition for the Chern class c_1 of a holomorphic principal $\boldsymbol{C}^*$-bundle $\xi = (E, B, \boldsymbol{C}^*, \pi)$. Let (U_j) be a covering by coordinate neighborhoods, z_j the coordinate of the standard fibre over U_j, (f_{jk}) the transition functions, (a_j) a cross section of the bundle with fibre $\boldsymbol{R}^+$ defined by the transition functions $f_{jk}\cdot\bar{f}_{jk}$. Then $c_1(\xi)$ is the class of the form $\gamma = (i/2\pi)\partial\bar{\partial}\log a_j$. *This class is equal to* ${}^1c_1$.

Proof. We have $\pi^*(\gamma) = d\psi$, where ψ is a 1-form over E with the local

representation $(-i/2\pi)\bar{\partial}\log(\bar{z}_j/a_j)$ over U_j; the restriction of ψ to a fibre is $(-i/2\pi)\bar{z}_j^{-1}d\bar{z}_j$ and its integral over the positively oriented unit circle is -1. Thus the Kodaira class is equal to $-\tau(x)$, where x is the canonical generator of $H^1(\boldsymbol{C}^*,\boldsymbol{Z})$, and is equal to ${}^1c_1$ by 29.4.

(d) On a complex manifold B of complex dimension n, the *canonical bundle* K is the line bundle of exterior forms of type $(n,0)$, i.e. the bundle of n-forms on the tangent bundle. By (10.6a, b), its Chern class is $-{}^1c_1(\theta)$, where θ is the tangential bundle. In particular, the determinant g of a positive non-degenerate hermitian metric provides a section of the bundle with fibre $\boldsymbol{R}^+$ and transition functions $|f_{jk}|^2$, (f_{jk} being the transition functions of K). Thus the Ricci form

$$(-i/2\pi)\partial\bar{\partial}\log g,$$

where $R_{\alpha\bar{\beta}} = -\partial^2\log g/\partial z_\alpha\partial\bar{z}_\beta$ is the Ricci tensor, represents ${}^1c_1(\theta)$ (see Kodaira, Annals of Math. 60 (1954), 28-48).

Appendix II.

30. Pontrjagin classes.

30.1. *Notation.* Tors A is the torsion subgroup of the commutative group A, and $\mathrm{Tors}_p A$ its p-primary component.

Let V be a vector space graded by finite dimensional subspaces V^i $(i \geqq 0)$. By $P(V,t)$, we denote its Poincaré polynomial

$$P(V,t) = \sum \dim V^i \cdot t^i,$$

and for a topological space X, we write $P_p(X,t)$ for $P(H^*(X,\boldsymbol{Z}_p),t)$.

f^*_p and f^*_Z denote the homomorphism of cohomology rings over $\boldsymbol{Z}_p$ and $\boldsymbol{Z}$ induced by a continuous map f.

Let ξ be a bundle with connected fibres, and A a commutative group. Then $T^i(F_\xi,A)$ or simply $T^i{}_\xi$ denotes the subgroup of transgressive elements in $H^i(F_\xi,A)$. We recall that the transgression τ_ξ is a homomorphism of $T^i{}_\xi$ into the quotient of $H^{i+1}(B_\xi,A)$ by a subgroup which will be denoted by $L^{i+1}(B_\xi,A)$ or $L^{i+1}{}_\xi$; we have $L^2{}_\xi = 0$.

30.2. *Cohomology* mod p *of* $B_{\boldsymbol{O}(n)}$ *and* $B_{\boldsymbol{SO}(n)}$. Let G be a compact Lie group, T a maximal torus. The ring of invariants of $W(G)$ operating in the usual way in $H^*(B_T,\boldsymbol{Z})$ is denoted by I_G. If $G = \boldsymbol{SO}(2n+1)$, $\boldsymbol{O}(2n+1)$, $\boldsymbol{O}(2n)$, (respectively $G = \boldsymbol{SO}(2n)$), and if (y_j) is the base induced by trans-

gression in (E_G, B_T, T) from the basis of $H^1(T, \boldsymbol{Z})$ discussed in § 9, then $W(G)$ is the group of permutations of the y_j's combined with an arbitrary (respectively even) number of changes of signs, and consequently $I_G = S(y_1^2, \cdots, y_n^2)$, (respectively is generated by $S(y_1^2, \cdots, y_n^2)$ and by the product of the y_i's).

PROPOSITION. *Let $G = \boldsymbol{SO}(n)$ or $\boldsymbol{O}(n)$, let $\boldsymbol{T}$ be its standard maximal torus, and let $\boldsymbol{Q}$ be the subgroup of diagonal matrices. Then*

(a) *For $p \neq 2$, $\rho^*_p(\boldsymbol{T}, G)$ maps $H^*(B_G, \boldsymbol{Z}_p)$ isomorphically onto $I_G \otimes \boldsymbol{Z}_p$.*

(b) *$\rho^*_2(\boldsymbol{Q}, \boldsymbol{O}(n))$ maps $H^*(B_{\boldsymbol{O}(n)}, \boldsymbol{Z}_2)$ isomorphically onto $S(u_1, \cdots, u_n)$, where (u_i) is a suitable basis of $H^1(B_{\boldsymbol{Q}}, \boldsymbol{Z}_2)$.*

For (b), see [3, Théorème 5], where a similar statement is also proved for $\boldsymbol{SO}(n)$. For $G = \boldsymbol{SO}(n)$, the assertion (a) is proved in [2, § 29]. For $G = \boldsymbol{O}(n)$, it follows from the more general

30.3. PROPOSITION. *Let G be a compact Lie group, G_0 its largest connected subgroup, T a maximal torus. If $H^*(G_0, \boldsymbol{Z})$ has no p-torsion and if the order of G/G_0 is not divisible by p, then $\rho^*_p(T, G)$ is an isomorphism of $H^*(B_G, \boldsymbol{Z}_p)$ onto $I_G \otimes \boldsymbol{Z}_p$.*

For $p = 0$, see [2, Prop. 27.1]. For p prime, the proof is practically identical, in view of the absence of torsion on G_0/T, (14.2), and is left to the reader.

30.4. *A remark on the Bockstein homomorphism.* Let X be a space with finitely generated integral cohomology groups. Let r_i be the number of cyclic direct summands of the p-primary component of $H^i(X, \boldsymbol{Z})$. Then by the universal coefficient formula

$$P_p(X, t) - P_0(X, t) = (1 + 1/t) \sum r_i \cdot t^i.$$

Let β_p be the Bockstein homomorphism attached to the exact sequence

$$0 \to \boldsymbol{Z} \xrightarrow{\alpha} \boldsymbol{Z} \to \boldsymbol{Z}_p \to 0 \qquad (\alpha(x) = px, x \in \boldsymbol{Z})$$

followed by reduction mod p. Clearly $\beta_p \circ \beta_p = 0$, and

$$s_i = \dim \beta_p(H^{i-1}(X, \boldsymbol{Z}_p))$$

is the number of torsion coefficients of $\mathrm{Tors}_p H^i(X, \boldsymbol{Z})$ which are equal to p. Thus we see:

LEMMA. $\mathrm{Tors}_p H^*(X,\mathbf{Z})$ *consists of elements of order* p *if and only if*

$$P_p(X,t)-P_0(X,t)=(1+1/t)P(\beta_p(H^*(X,\mathbf{Z}_p)),t).$$

When this is the case, the kernel of β_p *is the reduction* $\mathrm{mod}\, p$ *of* $H^*(X,\mathbf{Z})$, *and its image is the reduction* $\mathrm{mod}\, p$ *of* $\mathrm{Tors}_p H^*(X,\mathbf{Z})$.

We recall that β_p is an antiderivation and that $\beta_2=Sq^1$.

30.5. *Integral cohomology of* $B_{\mathbf{O}(n)}$ *and* $B_{\mathbf{SO}(n)}$. In this section, we shall prove that the torsion elements of the cohomology of these classifying spaces are all of order 2; first we insert a remark to be used in the proof.

Let H be an anticommutative graded algebra with unit over a field K, with $H^0=K$, and let D be an antiderivation on H, raising degrees by one, of square zero. Let A be a graded subspace stable under D. We denote by N_A, M_A, I_A, J_A, respectively, the kernel of D in A, a graded supplementary subspace to N_A in A, the image of A under D, a graded supplementary subspace to I_A in N_A. Since D increases degrees by one and is an isomorphism of M_A onto I_A, we have

$$P(A,t)=(1+1/t)P(I_A,t)+P(J_A,t). \tag{1}$$

Let now B be a second graded subspace stable under D, linearly disjoint from A over K; i.e., such that the subspace $A\cdot B$ spanned by the products $a\cdot b$ $(a\in A, b\in B)$ is isomorphic to $A\otimes B$ under the natural map $a\otimes b\to a\cdot b$. Using the previous notations, we have, as an elementary special case of the Künneth formula

$$P(J_{A\cdot B},t)=P(J_A,t)\cdot P(J_B,t). \tag{2}$$

PROPOSITION. *The torsion elements of* $H^*(B_{\mathbf{O}(n)},\mathbf{Z})$ *and* $H^*(B_{\mathbf{SO}(n)},\mathbf{Z})$ *are of order 2.*

It follows from 30.2 that these spaces have only 2-torsion. By 30.4, there remains to prove that for $G=\mathbf{SO}(n),\mathbf{O}(n)$, we have

$$P_2(B_G,t)-P_0(B_G,t)=(1+1/t)P(Sq^1(H^*(B_G,\mathbf{Z}_2)),t). \tag{3}$$

We have

$$\begin{aligned} &H^*(B_{\mathbf{SO}(n)},\mathbf{Z}_2)=\mathbf{Z}_2[w_2,\cdots,w_n],\ Sq^1w_i=(i+1)w_{i+1}\\ &H^*(B_{\mathbf{O}(n)},\mathbf{Z}_2)=\mathbf{Z}_2[w_1,\cdots,w_n],\ Sq^1w_i=w_1w_i+(i+1)w_{i+1}\end{aligned} \tag{4}$$

($w_i=i$-th Stiefel-Whitney class, $i\leqq n, w_{n+1}=0$), (see e.g. [3]). We first consider $\mathbf{SO}(n)$. By the foregoing, we may write

$$H^*(B_{SO(2m+1)}, \mathbf{Z}_2) \cong A_1 \otimes \cdots \otimes A_m,$$

$$H^*(B_{SO(2m)}, \mathbf{Z}_2) \cong A_1 \otimes \cdots \otimes A_{m-1} \otimes A'_m,$$

where

$$A_i = \mathbf{Z}_2[w_{2i}, w_{2i+1}] \quad (1 \leqq i \leqq m), \quad A'_m = \mathbf{Z}_2[w_{2m}]$$

are stable under the cup product and Sq^1, and A'_m is annihilated by Sq^1. By (4), the kernel of Sq^1 in A_i is spanned by the elements

$$w_{2i}^s \cdot w_{2i+1}^t \qquad (s \geqq 0, s \text{ even}, t \geqq 0),$$

and its image by the elements

$$w_{2i}^s \cdot w_{2i+1}^t \qquad (s \text{ even}, t > 0).$$

Thus, in the notations above, we may take as base for M_{A_i} (respectively J_{A_i}), the monomials

$$w_{2i}^s \cdot w_{2i+1}^t, \quad (s \text{ odd}, t \geqq 0), \quad (\text{respectively } w_{2i}^s, s = 0, 2, 4, 6, \cdots)$$

and we obtain

$$P(J_{A_i}, t) = (1 - t^{4i})^{-1}, \qquad (1 \leqq i \leqq m)$$

and, since $Sq^1(A'_m) = 0$

$$P(J_{A'_m}, t) = P(A'_m, t) = (1 - t^{2m})^{-1}.$$

By iterated application of (2) and by (1), we get

$$P_2(B_{SO(2m+1)}, t) = (1 + 1/t) P(Sq^1(H^*(B_{SO(2m+1)}, \mathbf{Z}_2)), t) + \prod_{i=1}^{m} (1 - t^{4i})^{-1},$$

$$P_2(B_{SO(2m)}, t) = (1 + 1/t) P(Sq^1(H^*(B_{SO(2m)}, \mathbf{Z}_2)), t) + (1 - t^{2m})^{-1} \prod_{i=1}^{m-1} (1 - t^{4i})^{-1},$$

which proves (3) for the unimodular case, since in both formulas, the last term on the right is the rational Poincaré series in view of 30.2.

We now pass to $\boldsymbol{O}(n)$. Choose a new basis of $H^*(B_{O(n)}, \mathbf{Z}_2)$ by

$$w^*_1 = w_1, w^*_{2i} = w_{2i}, \qquad (i \leqq [n/2]),$$

$$w^*_{2i+1} = w_{2i+1} + w_i \cdot w_{2i}, \qquad (i < [n/2]);$$

we have

$$H^*(B_{O(n)}, Z_2) = \mathbf{Z}_2[w^*_1, \cdots, w^*_n],$$

$$Sq^1 w^*_1 = (w^*_1)^2; Sq^1 w^*_{2i} = w^*_{2i+1} \qquad (i < [n/2]),$$

$$Sq^1 w^*_{2m} = w^*_1 \cdot w^*_{2m}, \qquad (n = 2m),$$

$$Sq^1 w^*_{2i+1} = Sq^1 Sq^1 \cdot w^*_{2i} = 0, \qquad (i \geqq 1).$$

We put

$$H^*(B_{O(2m+1)}, \mathbf{Z}_2) = A_0 \otimes A_1 \otimes \cdots \otimes A_m,$$
$$H^*(B_{O(2m)}, \mathbf{Z}_2) = A'_0 \otimes A_1 \otimes \cdots \otimes A_{m-1},$$

where

$$A_i = \mathbf{Z}_2[w^*_{2i}, w^*_{2i+1}] \qquad (1 \leqq i \leqq m),$$
$$A_0 = \mathbf{Z}_2[w^*_1] = \mathbf{Z}_2[w_1],$$
$$A'_0 = \mathbf{Z}_2[w^*_1, w^*_{2m}] = \mathbf{Z}_2[w_1, w_{2m}].$$

These are stable under cup-product and Sq^1, and we have as above

$$P(J_{A_i}, t) = (1 - t^{4i})^{-1}, \qquad (1 \leqq i \leqq m).$$

In A_0, the kernel of Sq^1 is spanned by $w_1{}^{2s}$ $(s \geqq 0)$; and the image by $w_1{}^{2s}$ $(s \geqq 1)$; hence $P(J_{A_0}, t) = 1$. Let us now prove that

$$P(J_{A'_0}, t) = (1 - t^{2m})^{-1}. \tag{5}$$

We have

$$Sq^1(w_1{}^s \cdot w_{2m}{}^t) = (s+t) w_1{}^{s+1} w_{2m}{}^t$$

which shows that the monomials $w_1{}^s w_{2m}{}^t$ with $s+t$ even (respectively $s+t$ even and $s > 0$) span $N_{A'_0}$ (respectively $I_{A'_0}$). Thus we may take the elements $w_{2m}{}^t$ (t even) as a base for $J_{A'_0}$, and this proves (5). The remainder of the proof of (3) for $\boldsymbol{O}(n)$ is then the same as for $\boldsymbol{SO}(n)$.

30.6. Corollary. *Let $G = \boldsymbol{O}(n)$ or $\boldsymbol{SO}(n)$. Then the kernel of Sq^1 on $H^*(B_G, \mathbf{Z})$ is the reduction* mod 2 *of $H^*(B_G, \mathbf{Z})$ and its image is the reduction* mod 2 *of* Tors $H^*(B_G, \mathbf{Z})$. *An element of $H^*(B_G, \mathbf{Z})$ is completely determined by its canonical images in $H^*(B_G, \mathbf{R})$ and $H^*(B_G, \mathbf{Z}_2)$.*

The first assertion follows from 30.4 and 30.5. The second one is an elementary fact about spaces with torsion elements of order 2 only.

In connection with the integral cohomology of $B_{O(n)}$ and $B_{SO(n)}$, let us also mention the following

30.7. Proposition. *Let $G = \boldsymbol{O}(n)$, $\boldsymbol{SO}(n)$ and let T be a maximal torus of G. Then $\rho^*_Z(T, G)$ maps $H^*(B_G, \mathbf{Z})$ onto I_G; its kernel is the torsion subgroup of $H^*(B_G, \mathbf{Z})$.*

We have seen in 9.3 that $S(y_1{}^2, \cdots, y_m{}^2)$, $(m = [n/2])$, is contained in $\rho^*_Z(T, G)$, which proves our statement for $G = \boldsymbol{SO}(2m+1)$, $\boldsymbol{O}(2m+1)$, $\boldsymbol{O}(2m)$. For $G = \boldsymbol{SO}(2m)$, we have to know moreover that $\rho^*_Z(T, \boldsymbol{SO}(2m))$ contains the product of the y_j's, but this follows from 9.5.

30.8. *Pontrjagin classes.* We follow the notations of 9.2, 9.3. By 30.6, the equalities

(6) $$\rho^*_0(T,G)(\tilde{p}) = \prod(1+y_i^2), \qquad (G = \boldsymbol{O}(n), \boldsymbol{SO}(n)),$$

(7) p_i (respectively $p_{i+\frac{1}{2}}$) reduced mod 2, is equal to w_{2i}^2 (respectively w_{2i+1}^2),

completely characterize the integral Pontrjagin classes. (6), over the integers, follows from 9.1, 9.3, 9.4. It implies that $p_{i+\frac{1}{2}}$ is a torsion element. (7) needs only to be proved for $G = \boldsymbol{O}(n)$ and, in view of 30.2(b), will follow from

(8) $$\rho^*_2(\boldsymbol{Q}(n), \boldsymbol{O}(n))(\tilde{p}) = \prod(1+u_j^2).$$

We have

$$\rho^*_2(\boldsymbol{Q}(n), \boldsymbol{O}(n))(\tilde{p}) = \rho^*_2(\boldsymbol{Q}(n), \boldsymbol{U}(n))(c) = \rho^*_2(\boldsymbol{Q}(n), \boldsymbol{T}')(\prod(1+x_i))$$

where $\boldsymbol{T}'$ is the subgroup of diagonal matrices of $\boldsymbol{U}(n)$ and (x_i) is the standard basis of $H^2(B_{T'}, \boldsymbol{Z})$, and therefore (8) follows from [4, § 11]:

$$\rho^*_2(\boldsymbol{Q}(n), \boldsymbol{T}')(x_i) = u_i^2 \qquad (1 \leqq i \leqq n).$$

30.9. *Remark on integral Stiefel-Whitney classes.* It follows also from 9.5, 30.6, (6), (7) that for an $\boldsymbol{SO}(2m)$-bundle, we have

(9) $$p_m = W_{2m}{}^2,$$

both sides being considered as integral classes. The relations $w_{2i+1} = Sq^1 w_{2i}$ for $\boldsymbol{SO}(n)$-bundles and (30.6), show that the universal w_{2i+1} is the reduction mod 2 of a uniquely determined element

$$W_{2i+1} \in H^{2i+1}(B_{\boldsymbol{SO}(n)}, \boldsymbol{Z})$$

of order 2, the *integral* $2i+1$*-th Stiefel-Whitney class,* and that we also have

$$p_{i+\frac{1}{2}} = (W_{2i+1})^2$$

over the integers.

30.10. *Pontrjagin classes and transgression.* As usual, $V_{n,k}$ denotes the Stiefel-manifold of orthonormal k-frames in euclidean n-space. We recall [2, § 10] that for n odd, $H^j(V_{n,n-2i+1}, \boldsymbol{Z})$ is equal to $\boldsymbol{Z}$ for $j = 0, 4i-1$, to $\boldsymbol{Z}_2$ for j even running from $2i$ to $4i-2$, and is zero for the other values of j which are $\leqq 4i-1$. We denote by v_i a generator of $H^{4i-1}(\boldsymbol{V}_{n,n-2i+1}, \boldsymbol{Z})$.

The first non-vanishing integral cohomology group of strictly positive dimension of the complex Stiefel manifold $\boldsymbol{W}_{n,n-2i+1}$ is of dimension $4i-1$ and is infinite cyclic. We denote its canonical generator (see 29.1) by t_i.

The inclusion of $\boldsymbol{O}(n)$ in $\boldsymbol{U}(n)$ induces a natural injective map of $\boldsymbol{V}_{n,k}=\boldsymbol{O}(n)/\boldsymbol{O}(n-k)$ into $\boldsymbol{W}_{n,k}=\boldsymbol{U}(n)/\boldsymbol{U}(n-k)$.

LEMMA. *Let n be odd, and $\lambda_{n,i}$ be the natural inclusion of $\boldsymbol{V}_{n,n-2i+1}$ in $\boldsymbol{W}_{n,n-2i+1}$. Let v_i and t_i be defined as above. Then $\lambda^*_{n,i}(t_i)=\pm 2v_i$.*

This lemma will be proved in the next section. Let ξ be a principal $\boldsymbol{O}(n)$-bundle, and ξ' be its complex extension. The given homomorphism of ξ into ξ' induces in a natural way a homomorphism $\alpha\colon \eta\to\eta'$ of the associated bundles with respective typical fibres $\boldsymbol{V}_{n,n-2i+1}$ and $\boldsymbol{W}_{n,n-2i+1}$. By the transgression definition (29.3(7)), we have $c_{2i}=-\tau_{\eta'}(t_i)$. Hence 29.4 and the lemma imply the

THEOREM. *Let n be odd and t_i be the canonical generator of $H^{4i-1}(\boldsymbol{W}_{n,n-2i+1},\boldsymbol{Z})$ and choose $v_i\in H^{4i-1}(\boldsymbol{V}_{n,n-2i+1},\boldsymbol{Z})$ such that $2v_i=\lambda^*_{n,i}(t_i)$. Let ξ be a principal $\boldsymbol{O}(n)$-bundle, η the associated bundle with fibre $\boldsymbol{V}_{n,n-2i+1}$. Then $p_i(\xi)=(-1)^{i+1}\tau_\eta(2v_i)$ modulo $L^{4i}{}_\eta$.*

For the notation $L^{4i}{}_\eta$, see 30.1. Since the cohomology groups of $\boldsymbol{V}_{n,n-2i+1}$ in positive dimensions $<4i-1$ are 2-groups, the spectral sequence definition of $L^{4i}{}_\eta$ [2, § 5] shows that $L^{4i}{}_\eta$ is a 2-group, so that the theorem characterizes p_i up to 2-torsion. We remark also that v_i itself is not universally transgressive (see following proof).

Let n be odd and β be the natural projection of E_ξ onto $E_\eta=E_\xi/\boldsymbol{O}(2i-1)$. Then we have of course

$$p_i(\xi)=(-1)^{i+1}\tau_\xi(\beta^*(2v_i))\quad \bmod L^{4i}{}_\xi.$$

By [2, § 10], $\beta^*(v_i)$ generates a direct summand of $H^{4i-1}(\boldsymbol{O}(n),\boldsymbol{Z})$ or of $H^{4i-1}(\boldsymbol{SO}(n),\boldsymbol{Z})$.

30.11. *Proof of the lemma.* We first consider the case where $n=2i+1$ and denote by ξ the universal bundle for $\boldsymbol{O}(2i+1)$, by ξ' its complex extension, by η and η' the associated bundles with fibres $\boldsymbol{V}_{2i+1,2}$ and $\boldsymbol{W}_{2i+1,2}$ and by α the natural map of η in η'. We have $H^*(\boldsymbol{V}_{2i+1,2},\boldsymbol{Z}_2)=\bigwedge(h_{2i-1},h_{2i})$, with $Sq^1h_{2i-1}=h_{2i}$ and $\tau_\eta(h_{2i-1})=w_{2i}$, $\tau_\eta(h_{2i})=w_{2i+1}$ (see for instance [3]); this implies easily that $h_{2i-1}\cdot h_{2i}$ is not universally transgressive. Since $h_{2i-1}\cdot h_{2i}$ is the reduction mod 2 of v_i, the latter is not universally transgressive in integral cohomology. For $\boldsymbol{V}_{2i+1,2}$, we have $H^0=H^{4i-1}=\boldsymbol{Z}$, $H^{2i}=\boldsymbol{Z}_2$ (see e.g. [2], § 10), therefore, the non-zero terms in the spectral sequence of η over the integers have fibre degrees 0, $2i$, $4i-1$ and those with

fibre degree $2i$ have order 2. Therefore $2t_i$ is universally transgressive and $L^{4i}{}_\eta$ consists of elements of order 2.

We know now that $2t_i$ generates $T^{4i-1}(\boldsymbol{V}_{2i+1,2}, \boldsymbol{Z})$; this group contains necessarily $\lambda^*{}_{2i+1,i}(v_i)$ since v_i is transgressive in η'. Thus we have $\lambda^*{}_{2i+1,i}(v_i) = 2k \cdot t_i$ for some integer k, hence also

$$p_i = \pm k\tau_\eta(2t_i) \quad \operatorname{mod} L^{4i}{}_\eta.$$

Let $\boldsymbol{T}$ be the standard maximal torus of $\boldsymbol{O}(2i+1)$. Then, in the notation of 9.3, we get from (1) in 9.3 and from the fact that $L^{4i}{}_\eta$ is a 2-group:

$$y_1{}^2 \cdot \cdots \cdot y_i{}^2 = \pm k\rho^*{}_{\boldsymbol{Z}}(\boldsymbol{T}, \boldsymbol{O}(2i+1))(\tau_\eta(2t_i)).$$

Since $H^*(B_{\boldsymbol{T}}, \boldsymbol{Z})$ is the ring of polynomials in the y_j's, this yields $k = \pm 1$, and the lemma for $n = 2i+1$.

In the general case, we consider the commutative diagram

$$\begin{tikzcd} \boldsymbol{V}_{n,n-2i+1} \arrow[r, "\lambda_{n,i}"] & \boldsymbol{W}_{n,n-2i+1} \\ \boldsymbol{V}_{2i+1,2} \arrow[u, "\mu"] \arrow[r, "\lambda_{2i+1,i}"] & \boldsymbol{W}_{2i+1,2}, \arrow[u, "\nu"'] \end{tikzcd}$$

where μ, ν are the injection of a fibre in the standard fiberings. It follows from §§ 9, 10 in [2] that μ^*, ν^* are isomorphisms in dimension $4i-1$; therefore the general case of the lemma follows by commutativity of the above diagram from the particular case already proved.

30.12. *A property of* $\rho^*{}_{\boldsymbol{Z}}(\boldsymbol{O}(r), \boldsymbol{O}(n))$. We end this appendix by proving that $\rho^*{}_{\boldsymbol{Z}}(\boldsymbol{O}(r), \boldsymbol{O}(n))$ is surjective ($r \leqq n$), a fact which is useful in the discussion of relative Pontrjagin classes (see M. Kervaire, Amer. J. Math. 79 (1957)).

LEMMA. *Let X, Y be two spaces with finitely generated integral cohomology groups and $f\colon X \to Y$ be a continuous map. We assume:*

(a) *The orders of the torsion elements of $H^*(X, \boldsymbol{Z})$ and $H^*(Y, \boldsymbol{Z})$ are square free.*

(b) *$f^*{}_0$ is surjective.*

(c) *For all primes p, $f^*{}_p$ is a surjective map for the kernels of the Bockstein maps β_p* (see 30.4).

Then $f^*{}_{\boldsymbol{Z}}$ is surjective.

Let M_i be the image of $H^i(Y,\boldsymbol{Z})$ under f^*. By (b), it has a finite index, say g_i, in $H^i(X,\boldsymbol{Z})$. In view of (30.4), the assumptions (a), (c) imply that

$$f^*: H^i(Y,\boldsymbol{Z})\otimes \boldsymbol{Z}_p \to H^i(X,\boldsymbol{Z})\otimes \boldsymbol{Z}_p \qquad (i \geqq 0, p \text{ prime})$$

is surjective, or in other words, that

$$H^i(X,\boldsymbol{Z}) = p\cdot H^i(X,\boldsymbol{Z}) + M_i,$$

hence, by iteration,

$$H^i(X,\boldsymbol{Z}) = g_i\cdot H^i(X,\boldsymbol{Z}) + M_i = M_i.$$

PROPOSITION. *The homomorphism* $\rho^*_{Z}(\boldsymbol{O}(r),\boldsymbol{O}(n))$, $(r \leqq n)$, *is surjective.*

By (30.5), $B_{\boldsymbol{O}(r)}$ and $B_{\boldsymbol{O}(n)}$ satisfy (a) of the lemma.

It follows from (9.3) and (30.2) that $H^*(B_{\boldsymbol{O}(n)},\boldsymbol{Z}_p)$ is the ring of polynomials in the universal Pontrjagin classes $(i \geqq 1)$ for $p \neq 2$. Since these are preserved under the natural inclusion $\boldsymbol{O}(r) \subset \boldsymbol{O}(n)$ (see 9.7), it follows that $\rho^*_p(\boldsymbol{O}(r),\boldsymbol{O}(n))$ is surjective for $p \neq 2$. This shows that (b) is fulfilled and also, in view of (30.2), (30.5), that (c) is true for $p \neq 2$. Thus, in order to derive the proposition from the lemma, there remains to show that

(10) $\rho^*_2(\boldsymbol{O}(r),\boldsymbol{O}(n))$ *is surjective for the kernels of* Sq^1.

We write ρ^* for $\rho^*_2(\boldsymbol{O}(r),\boldsymbol{O}(n))$, denote by w_i (respectively $\bar{w}_i$) the universal Stiefel-Whitney classes for $\boldsymbol{O}(r)$- (respectively $\boldsymbol{O}(n)$-) bundles and define w^*_i, $\bar{w}^*_i$ as in the proof of (3).

The assertion (10) is clearly true on any subalgebra $A \subset H^*(B_{\boldsymbol{O}(n)},\boldsymbol{Z}_2)$ which is stable under Sq^1 and mapped injectively by ρ^*; the latter is given by

$$\rho^*(\bar{w}_i) = w_i, \quad \rho^*(\bar{w}^*_i) = w^*_i, \quad (i \leqq r), \quad \rho^*(\bar{w}_j) = 0 \quad (j > r).$$

Therefore, by (4), this applies to

$$A = \boldsymbol{Z}_2[\bar{w}_1,\cdots,\bar{w}_i] \quad (i \leqq r, i \text{ odd}), \quad \boldsymbol{Z}_2[\bar{w}_1,\cdots,\bar{w}_{r-1},\bar{w}_r{}^2] \qquad (r \text{ even}),$$
$$\boldsymbol{Z}_2[\bar{w}_1{}^2, \bar{w}^*_2,\cdots,\bar{w}^*_{r-1},\bar{w}_r{}^2] \quad (r \text{ even}).$$

This establishes (10) for r odd; for $r = 2m$ even, it reduces its proof to that of the following statement: given

$$x = w_{2m}\cdot P + Q, Sq^1x = 0, (P,Q \in \boldsymbol{Z}_2[w_1,\cdots,w_{2m-1},w_{2m}{}^2]),$$

there exists $y \in H^*(B_{\boldsymbol{O}(n)},\boldsymbol{Z}_2)$ with the properties

$$Sq^1 y = 0, \qquad \rho^*(y) = x.$$

$Sq^1 x = 0$ gives

$$w_{2m}(w_1 P + Sq^1 P) + Sq^1 Q = 0$$

and, since $\mathbf{Z}_2[w_1, \cdots, w_{2m-1}, w_{2m}{}^2]$ is stable under Sq^1,

$$w_1 P + Sq^1 P = Sq^1 Q = 0.$$

We may write

$$P = w_1 R + S, \qquad (R, S \in \mathbf{Z}_2[w_1{}^2, w^*{}_2, \cdots, w^*{}_{2m-1}, w_{2m}{}^2].$$

Hence

$$Sq^1 P = w_1{}^2 R + w_1 \cdot Sq^1 R + Sq^1 S,$$

$$0 = w_1 P + Sq^1 P = w_1(S + Sq^1 R) + Sq^1 S,$$

and, as before,

$$S + Sq^1 R = Sq^1 S = 0.$$

Now let $\bar{P}, \bar{Q}, \bar{R} \in H^*(B_{O(n)}, \mathbf{Z}_2)$ be the elements obtained from P, Q, R by barring the w_i's. Then a trivial computation shows that

$$y = \bar{w}_{2m} \cdot \bar{P} + \bar{Q} + \bar{w}_{2m+1} \bar{R}$$

has the desired properties.

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EXTENSIONS OF PURE STATES.*

By Richard V. Kadison [1] and I. M. Singer.

1. Introduction and preliminaries. The main concern of this paper is the problem of uniqueness of extensions of pure states from maximal abelian self-adjoint algebras of operators on a Hilbert space to the algebra of all bounded operators on that space. The answer, as many of us have suspected for several years, is in the negative. To the best of our knowledge, the problem is not recorded in the literature. We heard of it first from I. E. Segal and I. Kaplansky, though it is difficult to credit a problem which stems naturally from the physical interpretation and the inherent structure of a subject. This problem has arisen, in one form or another, in our work on several different occasions; and we have been gathering bits of information related to it, over the years. (The present solution is prompted by just such a reappearance.)

To state the problem precisely, let $\mathcal{H}$ be a (complex) Hilbert space and $\mathfrak{A}$ an algebra of bounded operators invariant under the adjoint operation ($A \rightarrow A^*$), containing the identity operator, I, and closed in the uniform (operator bound) topology. The algebra, $\mathfrak{A}$, is a C^*-algebra, and a linear functional, f, on $\mathfrak{A}$ which is 1 at I and real, non-negative on positive operators (those operators, A, such that $(Ax, x) \geqq 0$ for each x in $\mathcal{H}$), is a state of $\mathfrak{A}$. The set of states of $\mathfrak{A}$ is a convex subset of the dual of $\mathfrak{A}$ and is compact in the w^*-topology on the dual (the weak topology induced by $\mathfrak{A}$). The Krein-Milman theorem tells us that the set of states is the closed convex hull of its extreme points—these are the pure states of $\mathfrak{A}$. An argument of the Hahn-Banach type enables us to extend states from a C^*-subalgebra of $\mathfrak{A}$ to $\mathfrak{A}$ [4]. The set of extensions of such a state forms a compact convex subset of the dual whose extreme points can easily be shown to be pure states of $\mathfrak{A}$ [4], provided that the state of the subalgebra is pure. Thus, if a pure state has a unique pure state extension from a C^*-subalgebra of a C^*-algebra to the algebra, then the closed convex hull of this extension, viz.

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itself, is the set of all state extensions of the given pure state. Thus a pure state of a C^*-subalgebra has a unique state extension to the algebra if and only if it has a unique pure state extension.

If the C^*-algebra $\mathfrak{A}$ is abelian, the set of pure states is compact, and the natural mapping of $\mathfrak{A}$ into the second dual, followed by the restriction mapping to the set of pure states, is an algebraic, isometric, order isomorphism of $\mathfrak{A}$ onto the algebra of continuous, complex-valued functions on this compact space (the pure state space). (This isomorphism carries the adjoint operation in $\mathfrak{A}$ into complex conjugation in the function space.) An easy Zorn's lemma argument shows that $\mathfrak{A}$ is contained in a maximal abelian, self-adjoint subalgebra, $\mathcal{A}$, of the algebra, $\mathcal{B}$, of all bounded operators on $\mathcal{H}$. (Making use of the decomposition of operators as a sum of a self-adjoint and a skew-adjoint operator, it is not difficult to show that $\mathcal{A}$ is maximal with respect to the property of being abelian.) The fact that bounded families of operators in $\mathcal{A}$ have least upper bounds causes the pure state space of $\mathcal{B}$ to be extremely disconnected (i.e., the closure of each open set is open [8]). Examples of maximal abelian, self-adjoint algebras arise from the multiplication algebras on $L_2(X,\mu)$, with μ a measure on the space X, i.e. the algebra of operators T_g, with g an essentially bounded, μ-measurable function on X, where $T_g(h) = gh$, for each h in $L_2(X,\mu)$. In particular, with X the unit interval and μ Lebesgue measure on X, the algebra, $\mathcal{A}_c$, which arises will be referred to as 'the continuous maximal abelian algebra'; and with X, the integers, and $\mu(n) = 1$, for all n, the algebra, $\mathcal{A}_d$, which arises will be referred to as 'the discrete algebra.' (The maximal abelian algebras on finite-dimensional spaces are constructed as $\mathcal{A}_d$ is, with the integers replaced by a finite set. The algebra, $\mathcal{A}_d$, can also be viewed as the set of bounded diagonal matrices relative to a complete orthonormal basis.) Each maximal abelian algebra on a separable Hilbert space is unitarily equivalent to $\mathcal{A}_c$, $\mathcal{A}_d$, a finite-dimensional maximal abelian algebra, or the direct sum of $\mathcal{A}_c$ with one of the last two types. The problem of extensions of pure states from maximal abelian algebras to $\mathcal{B}$ (we consider mainly the separable case throughout this paper) reduces then, to a study of extensions from $\mathcal{A}_c$ and $\mathcal{A}_d$. Although we refer to $\mathcal{A}_d$ as 'the discrete maximal abelian algebra,' it should be noted that there is a great deal of 'non-discreteness' about it. In fact, its pure state space is easily identified with the β-compactification of the integers [9].

Each unit vector, x, gives rise to a state of $\mathcal{B}$ by means of the mapping, $A \rightarrow (Ax, x)$; and it is easily seen that this state, ω_x, is a pure state of $\mathcal{B}$. We refer to ω_x as a 'vector state' (also 'discrete state'). From our previous

remarks, we see that a pure state of an abelian C^*-algebra is multiplicative (the converse is true for all C^*-algebras), and from this, that a vector state is pure on an abelian C^*-algebra if and only if the vector which induces it is an eigenvector for each operator in the algebra. (Note that two vector states of $\mathcal{B}$ are equal if and only if the two vectors differ by a scalar multiple of modulus 1.) Since some operators in $\mathcal{A}_c$ have no eigenvectors, none of the pure states of $\mathcal{A}_c$ is a vector state—yet, each such pure state has a pure state extension to $\mathcal{B}$. Thus, there are pure states of $\mathcal{B}$ which are not vector states—we call these 'singular pure states.' Indeed, the points of the β-compactification of the integers which do not correspond to integers give rise to pure states of $\mathcal{A}_d$ which are not vector states and, therefore, have singular pure state extensions to $\mathcal{B}$. A well-known fact about singular pure states (which can be read out of the results of [3], for example) tells us that the singular pure states are precisely those which annihilate all completely continuous operators.

The uniqueness problem for extensions of pure states is the following: is there a unique state extension of a pure state of a maximal abelian self-adjoint algebra of the algebra, $\mathcal{B}$, of all bounded operators to $\mathcal{B}$? There are, of course, the two subdivisions of this problem—the question for $\mathcal{A}_d$ and the question for $\mathcal{A}_c$. It is quite easy to see that uniqueness of extension cannot be expected for abelian C^*-algebras other than the maximal abelian ones. In fact, if $\mathfrak{A}$ is an abelian C^*-algebra and $\mathcal{A}$ is a maximal abelian one containing it properly, then, making use of the function representation (of $\mathcal{A}$), the Stone-Weierstrass theorem assures us that $\mathfrak{A}$ does not separate pure states of $\mathcal{A}$, i.e., that there are two distinct pure states of $\mathcal{A}$ which agree on $\mathfrak{A}$ (and this restriction to $\mathfrak{A}$, being multiplicative, is pure). Naturally, one wonders why uniqueness should be expected in the maximal abelian case. Classically, our maximal abelian algebra, $\mathcal{A}$, would be that associated with an orthonormal basis, viz. $\mathcal{A}_d$, and the pure state, one due to a basis vector, x. Since the one-dimensional projections on the basis elements lie in $\mathcal{A}_d$, another pure state, ρ, which agrees with ω_x on $\mathcal{A}_d$ will annihilate all these projections with the exception of the one whose range contains x; so that if ρ is a vector state, that vector differs from x by a scalar multiple of modulus 1. Thus $\rho=\omega_x$, when we note that if ρ were not a vector state, it would annihilate all completely continuous operators and, in particular, all one-dimensional projections. More generally, if $\mathcal{A}$ is a maximal abelian algebra, ω_x is a pure state of $\mathcal{A}$, E is the one-dimensional projection whose range contains x, and ω is a pure state extension of ω_x from $\mathcal{A}$ to all bounded operators, then $\omega=\omega_x$.

In fact, x is an eigenvector for each operator in $\mathcal{A}$, so that $\mathcal{A}$ leaves the range of E invariant and, hence, commutes with E (since $\mathcal{A}$ is a self-adjoint algebra). Thus E lies in $\mathcal{A}$, and ω is a vector state ω_y, since $\omega(E)=\omega_x(E)=1\neq 0$. From $\omega_y(E)=\omega_x(E)=1$, we conclude that $\|E\|=1$ and that y lies in the range of E. Being a unit vector, y is a scalar multiple of modulus 1 of x, and $\omega_y=\omega_x$, on all bounded operators.

Having these results for vector pure states, it is not unreasonable to expect them to hold for arbitrary pure states in the same way that one passes from certain properties of the point spectrum to those of the general spectrum. Indeed, a casual handling of limit processes (just allowing oneself the minor luxury of a sequential limit in place of a directed limit) leads to a "proof" of the uniqueness of pure state extensions—false but provocative. Add to this evidence the elusiveness of a counter-example and one has the case for the conjecture.

In [5], von Neumann introduces a process for taking the "diagonal part" of certain operators in a von Neumann algebra (strongly closed C^*-algebra) relative to a maximal abelian self-adjoint subalgebra. Among other things, this process is linear, order preserving, and idempotent, and, so, provides a continuous way of simultaneously extending all the pure states of a maximal abelian algebra (provides a cross-section in the sheaf-like structure of state extensions over the pure state space of the maximal abelian algebra, so to speak). Two distinct diagonal processes will, of course, settle the pure state extension problem negatively for a particular maximal abelian algebra. In Section 2, we give a discussion of diagonal processes suitable for our applications, and in 3, we prove the uniqueness of diagonal processes for $\mathcal{A}_d$ and the non-uniqueness of diagonal processes for $\mathcal{A}_c$. The non-uniqueness proof is a mixture of abstract and classical techniques which produces a specific operator with distinct "diagonal parts" relative to $\mathcal{A}_c$ (and, so, to which some pure state of $\mathcal{A}_c$ can be extended in more than one way). In the last section, we discuss related questions concerning pure states.

2. Diagonal processes. The lemma which follows provides the means for constructing diagonal processes relative to maximal abelian algebras.

LEMMA 1. *If $\mathcal{A}$ is an abelian von Neumann algebra generated by the projections $\{E\}_{n\in\mathcal{J}}$, $\mathcal{J}$ the set of positive integers, and p is a point of $\beta(\mathcal{J})-\mathcal{J}$, where $\beta(\mathcal{J})$ is the β-compactification of $\mathcal{J}$, then there is a linear operator, $\mathfrak{D}_p$, whose domain is the set of bounded operators and which is such that:*

(a) $\mathcal{D}_p(B)$ *commutes with each* E_n *(so, lies in* $\mathcal{A}'$*, the commutant of* $\mathcal{A}$*), and* $\mathcal{D}_p(B)$ *is a weak closure point of operators* $\{B^{|E_1|E_2|\cdots|E_n}\}$*, where* $T^{|E}$ *is* $ETE+(I-E)T(I-E)$.

(b) $\mathcal{D}_p(AB)=A\mathcal{D}_p(B)$*, for each* A *in* $\mathcal{A}$ *(and* $\mathcal{D}_p(BA)=\mathcal{D}_p(B)A$*).*

(c) $\mathcal{D}_p(I)=I$*, and* $\mathcal{D}_p(B)\geqq 0$ *if* $B\geqq 0$.

Proof. Note that

$$\|B^{|E}\|=\|EBE+(I-E)B(I-E)\|$$
$$=\max\{\|EBE\|,\|(I-E)B(I-E)\|\}\leqq\|B\|,$$

so that the function, f, defined on $\mathcal{I}$ by $f(n)=B^{|E_1|\cdots|E_n}$ maps $\mathcal{I}$ into the (weakly compact) ball of radius $\|B\|$ about 0 in the set of bounded operators. Note also that $B^{|E|F}=B^{|F|E}$ when $EF=FE$ and that $E(B^{|E})=(B^{|E})E$; so that $f(n)$ commutes with $E_1,\cdots,E_n$. From the properties of $\beta(\mathcal{I})$, we have that f has a unique extension, f_1, from $\mathcal{I}$ to $\beta(\mathcal{I})$ which is continuous and whose range is contained in the ball of radius $\|B\|$ about 0. We define $\mathcal{D}_p(B)$ to be $f_1(p)$. The observation that $(\alpha B+C)^{|E}=\alpha(B^{|E})+C^{|E}$, together with the fact that $\mathcal{I}$ is dense in the Hausdorff space $\beta(\mathcal{I})$, yields the linearity of $\mathcal{D}_p$ and the fact that $\mathcal{D}_p(B)$ is a weak closure point of $\{B^{|E_1|\cdots|E_n}\}$. If B is I, then $B^{|E_1|\cdots|E_n}=I$, for all n, so that $\mathcal{D}_p(B)=I$; and if $B\geqq 0$, then $B^{|E_1|\cdots|E_n}\geqq 0$, for all n, whence each weak closure point of $\{B^{|E_1|\cdots|E_n}\}$ is positive and, in particular, $\mathcal{D}_p(B)\geqq 0$. Moreover, $(AB)^{|E_k}=A(B^{|E_k})$ (and $(BA)^{|E_k}=(B^{|E_k})A$), with A in $\mathcal{A}$, so that $\mathcal{D}_p(AB)=A\mathcal{D}_p(B)$ (and $\mathcal{D}_p(BA)=\mathcal{D}_p(B)A$) (recall that left and right multiplication by A is weakly continuous). For a given n_0, $\mathcal{D}_p(B)$ is a weak closure point of $B^{|E_1|\cdots|E_m}$, with $m\geqq n_0$, each of which commutes with E_{n_0}. Thus $\mathcal{D}_p(B)$ commutes with E_{n_0}, for each n_0; so that $\mathcal{D}_p(B)$ lies in $\mathcal{A}'$.

DEFINITION 1. *A linear order preserving mapping from all bounded operators into the commutant of an abelian von Neumann algebra,* $\mathcal{A}$*, which is the identity on* $\mathcal{A}$ *is a "diagonal process relative to* $\mathcal{A}$*." If the image of each operator,* B*, is a weak closure point of operators* $B^{|E_1|\cdots|E_n}$*, with* $E_1,\cdots,E_n$ *in* $\mathcal{A}$*, the diagonal process is "proper"; otherwise, it is "improper."*

Remark 1. If $\mathcal{D}$ is proper and $B\in\mathcal{A}'$, $\mathcal{D}(B)$ is a weak closure point of $B^{|E_1|\cdots|E_n}=B$, so that $\mathcal{D}(B)=B$.

LEMMA 2. *If* $\mathcal{D}$ *is a diagonal process relative to* $\mathcal{A}$ *then* $\mathcal{D}(AB)$

$= A\mathfrak{D}(B)$ (*and* $\mathfrak{D}(BA) = \mathfrak{D}(B)A$) *for each* A *in* $\mathfrak{A}$ *and each bounded operator,* B. *If* $\mathfrak{D}$ *is weakly continuous on the unit ball, then it is the unique proper diagonal process relative to* $\mathfrak{A}$, *and* $\mathfrak{D}(B)$ *is the weak limit of* $\{B_n\}$, *where* $B_n = B^{|E_1|\cdots|E_n}$ *and* $\{E_n\}$ *is a generating family of projections for* $\mathfrak{A}$.

Proof. We remark first, that if T and S are distinct operators in $\mathfrak{A}'$, there is an extension to $\mathfrak{A}'$ of some pure state of $\mathfrak{A}$ (in fact, a pure state extension) which differs on T and S. In fact, for each cardinal, n, there is a projection P_n in $\mathfrak{A}$ such that $\mathfrak{A}P_n$ is an n-fold copy of some maximal abelian algebra, $\mathfrak{A}_n$, acting on a Hilbert space, $\mathcal{H}_n$ (i.e. $\mathfrak{A}'P_n$ acting on $P_n(\mathcal{H})$ is unitarily equivalent to the algebra of $n \times n$ matrices with entries in $\mathfrak{A}_n$ acting on the direct sum of $\mathcal{H}_n$ with itself n times, in the usual way), and $\sum_n P_n = I$. With T and S distinct, $TP_n \neq SP_n$, for some n. If we can establish our result for $\mathfrak{A}P_n$ and its commutant $\mathfrak{A}'P_n$ (on $P_n(\mathcal{H})$), there is a pure state ρ_n of $\mathfrak{A}P_n$ and a state extension, ρ_n', of it to $\mathfrak{A}'P_n$ such that $\rho_n'(TP_n) \neq \rho_n'(SP_n)$. Defining ρ and ρ' by $\rho(A) = \rho_n(AP_n)$ and $\rho'(A') = \rho_n'(A'P_n)$, respectively, we note that ρ, being multiplicative, is a pure state of $\mathfrak{A}$, ρ' is a state extension of it and $\rho'(T) \neq \rho'(S)$. We may assume therefore that $\mathfrak{A}$ is an n-fold copy of the maximal abelian algebra $\mathfrak{A}_0$ acting on $\mathcal{H}_0$; from which $\mathfrak{A}'$ is the algebra of all $n \times n$ matrices, with entries in $\mathfrak{A}_0$, which give bounded operators acting upon the direct sum of $\mathcal{H}_0$ with itself n times. Let X be the pure state space of $\mathfrak{A}_0$, so that $\mathfrak{A}_0$ is algebraically isomorphic to the algebra, $C(X)$, of complex-valued continuous functions on X. Some entry in the matrix representations of T and S are distinct and, so, differ at a pure state ρ_0 of $\mathfrak{A}_0$. Let $\bar{\rho}_0(A')$ be the matrix obtained by replacing each entry of A' by its value at ρ_0, for A' in $\mathfrak{A}'$. The operator corresponding to this matrix is bounded and positive if A' is positive. Indeed, with n finite, the boundedness is automatic and the positivity then follows from the fact that $\bar{\rho}_0$ is adjoint-preserving and multiplicative, since ρ_0 is. (Boundedness and positivity can also be established when ρ_0 is not assumed pure by making use of [1].) Thus $\|\bar{\rho}_0(A')\| \leqq \|A'\|$, when n is finite. Applying this to the infinite case, we see that each finite minor has norm not exceeding $\|A'\|$, and again $\|\bar{\rho}_0(A')\| \leqq \|A'\|$. As in the finite case, it now follows that $\bar{\rho}_0(A')$ is positive if A' is. Since $\bar{\rho}_0(T) \neq \bar{\rho}_0(S)$, there is a unit vector, x_0, in the Hilbert space direct sum of the complex numbers with itself n times such that $(\bar{\rho}_0(T)x_0, x_0) \neq (\bar{\rho}_0(S)x_0, x_0)$. Now $\bar{\rho}_0(A)$ is a scalar multiple of I, for each A in $\mathfrak{A}$ so that $A \to (\bar{\rho}_0(A)x_0, x_0)$ is a pure state, ρ, of $\mathfrak{A}$ and $A' \to (\bar{\rho}_0(A')x_0, x_0)$ is an extension, ρ', of it to $\mathfrak{A}'$. By construction,

$\rho'(T) \neq \rho'(S)$. (The set of all state extensions of ρ to $\mathcal{A}'$ is a compact convex subset of the set of states of $\mathcal{A}'$ whose extreme points are pure states of $\mathcal{A}'$. If T and S coincide on each of these pure state extensions of ρ, they coincide on their finite convex combinations, so, on their (w^*-) closed convex hull, i.e., on all state extensions of ρ—in particular, on ρ'. Thus T and S differ on some pure state extension to $\mathcal{A}'$ of a pure state of $\mathcal{A}$.)

If ρ is a pure state of $\mathcal{A}$ and ρ' a state extension of ρ to all bounded operators, then $\rho'(AB) = \rho'(A)\rho'(B)$ (and $\rho'(BA) = \rho'(B)\rho'(A)$), for each A in $\mathcal{A}$. In fact, if E is a projection in $\mathcal{A}$, $\rho'(E) = 0$ or 1, since ρ' is multiplicative on $\mathcal{A}$. Thus $\rho'(EB)$ or $\rho'[(I-E)B]$ is 0 (as $\rho'(E)$ is 0 or 1, respectively), by an application of Schwarz's inequality to the inner product $K, H \to \rho'(H^*K)$ on the algebra of bounded operators. In either case, $\rho'(EB) = \rho'(E)\rho'(B)$. Thus $\rho'(AB) = \rho'(A)\rho'(B)$ for operators A in $\mathcal{A}$ which are linear combinations of projections in $\mathcal{A}$, and, by continuity of ρ' in the uniform topology, for uniform limits of such operators. From the spectral theorem, each self-adjoint operator in $\mathcal{A}$ is such a limit, so that $\rho'(AB) = \rho'(A)\rho'(B)$, for each A in $\mathcal{A}$ and each bounded operator, B.

In particular, if ρ'' is a state extension of ρ to $\mathcal{A}'$ and $\rho' = \rho'' \circ \mathcal{D}$, then ρ' is a state extension of ρ to all bounded operators. (Recall that $\mathcal{D}$ is the identity transform on $\mathcal{A}$.) Thus,

$$\begin{aligned}\rho''(\mathcal{D}(AB)) &= \rho'(AB) = \rho'(A)\rho'(B) = \rho'(A)\rho''(\mathcal{D}(B)) \\ &= \rho''(A)\rho''(\mathcal{D}(B)) = \rho''(A\mathcal{D}(B)),\end{aligned}$$

with A in $\mathcal{A}$ and B a bounded operator. (Note that the last equality follows from the considerations of the preceding paragraph applied to an extension of ρ'' from $\mathcal{A}'$—and hence of ρ from $\mathcal{A}$—to all bounded operators.) Since $\mathcal{D}(AB)$ and $A\mathcal{D}(B)$ are in $\mathcal{A}'$ and ρ'' is an arbitrary state extension to $\mathcal{A}'$ of an arbitrary pure state of $\mathcal{A}$, $\mathcal{D}(AB) = A\mathcal{D}(B)$, from the results of the first paragraph of this proof.

Suppose, now, that $\mathcal{D}$ is weakly continuous on the unit ball (and, so, on each bounded ball), and that $\mathcal{D}'$ is a proper diagonal process relative to $\mathcal{A}$. In this case, $\mathcal{D}'(B)$ is a weak closure point of $\{B^{|E_1|\cdots|E_n}\}$. Each such weak closure point, A', is such that $\mathcal{D}(A')$ is a weak closure point of $\{\mathcal{D}(B^{|E_1|\cdots|E_n})\} = \{\mathcal{D}(B)\}$, whence $\mathcal{D}(A') = \mathcal{D}(B)$. With A' in $\mathcal{A}'$, $\mathcal{D}(A') = A'$, since $\mathcal{D}$ is proper (cf. Remark 1). Thus $\mathcal{D}'(B) = \mathcal{D}(B)$ and $\mathcal{D}' = \mathcal{D}$. Moreover, each weak limiting point, A', of the sequence $(B^{|E_1|\cdots|E_n})$, lies in $\mathcal{A}'$, since it commutes with each E_n, and is a weak closure point of $\{B^{|E_1|\cdots|E_n}\}$, so that $A' = \mathcal{D}(B)$. Since $\{B^{|E_1|\cdots|E_n}\}$ is

contained in the weakly compact ball of radius $\| B \|$ about 0, $(B^{|E_1|\cdots|E_n})$ has a weak limiting point which must be $\mathcal{D}(B)$. Thus $\mathcal{D}(B)$ is the weak limit of $(B^{|E_1|\cdots|E_n})$.

The next lemma notes the possibility of extending a positive linear mapping from a linear space of bounded operators containing I into an abelian von Neumann algebra to such mappings of all bounded operators into the abelian von Neumann algebra. The proof is a direct copy of the proof of Krein's extension theorem for states [4] making use of the boundedly complete lattice properties of abelian von Neumann algebras.

LEMMA 3. *If $\mathfrak{A}$ is an abelian von Neumann algebra, $\mathfrak{L}_0$ a self-adjoint linear space of bounded operators containing I, and ϕ_0 an adjoint-preserving, positive, linear mapping of $\mathfrak{L}_0$ into $\mathfrak{A}$, then ϕ_0 has an adjoint-preserving, positive linear extension with range in $\mathfrak{A}$ to the algebra, $\mathfrak{B}$, of all bounded operators.*

Proof. Partially order the set of adjoint-preserving, positive, linear mappings with range in $\mathfrak{A}$, domain a self-adjoint linear subspace of $\mathfrak{B}$ containing $\mathfrak{L}_0$, and which extend ϕ_0, by "function extension". Zorn's lemma applies, and there exists a maximal element, ϕ, with domain $\mathfrak{L}$. If $\mathfrak{L} \neq \mathfrak{B}$, there is a self-adjoint operator B not in $\mathfrak{L}$. Choose a positive integer n, such that $nI \geqq B \geqq -nI$. Then $-n\phi(I)$ is a lower bound for the subset $\{\phi(A) : A \in \mathfrak{L} \text{ and } A \geqq B\}$ of $\mathfrak{A}$ and $n\phi(I)$ is an upper bound for $\{\phi(C) : C \in \mathfrak{L} \text{ and } C \leqq B\}$. These subsets have a greatest lower bound, A_1, and least upper bound, A_0, respectively, in $\mathfrak{A}$, since $\mathfrak{A}$ is a boundedly complete lattice. Since $\phi(A) \geqq \phi(C)$, when $A \geqq B \geqq C$, with A and C in $\mathfrak{L}$, $A_1 \geqq A_0$. Choose A in $\mathfrak{A}$ such that $A_1 \geqq A \geqq A_0$, and define ϕ' on the linear space generated by B and $\mathfrak{L}$ as follows: $\phi'(\alpha B + C) = \alpha A + \phi(C)$, with C in $\mathfrak{L}$. Then ϕ' is an adjoint-preserving linear mapping with range in $\mathfrak{A}$ and is an extension of ϕ. If $\alpha B + C \geqq 0$, making use of the choice of A in each of the cases, $\alpha = 0$, $\alpha > 0$, $\alpha < 0$, we conclude that ϕ' is a positive mapping. Since $B \notin \mathfrak{L}$, the existence of ϕ' contradicts the maximality of ϕ. Thus $\mathfrak{L} = \mathfrak{B}$ and ϕ is an adjoint-preserving, positive, linear extension from $\mathfrak{L}$ to $\mathfrak{B}$ of ϕ_0.

Remark 2. With the notation of the preceding lemma, ϕ_0 has a unique positive extension from $\mathfrak{L}_0$ to $\mathfrak{B}$ if and only if the greatest lower bound of $\{\phi_0(A) : A \text{ in } \mathfrak{L}_0 \text{ and } A \geqq B\}$ is equal to the least upper bound of $\{\phi_0(C) : C \text{ in } \mathfrak{L}_0 \text{ and } B \geqq C\}$, for each self-adjoint operator, B, in $\mathfrak{B}$. In fact, if they are equal, the positivity condition forces each positive extension of ϕ_0 to take this value at B; and if they are not equal for some self-adjoint B, we may

extend ϕ_0 to the space generated by B and $\mathfrak{L}_0$ by assigning either of these values to B, and then extend the resulting mappings to two distinct positive mappings of $\mathcal{B}$ into $\mathcal{A}$, each of which extends ϕ_0.

Remark 3. If we take $\mathcal{A}$ as $\mathfrak{L}_0$ and ϕ_0 as the identity mapping on $\mathcal{A}$, the extension lemma guarantees the existence of a diagonal process with range in $\mathcal{A}$, and this diagonal process is improper if $\mathcal{A}$ is not maximal abelian (for a proper process is the identity on $\mathcal{A}'$, the commutant of $\mathcal{A}$). In case $\mathcal{A}$ is maximal abelian, and we proceed as just noted, the extension lemma and the the preceding remark provide another criterion for uniqueness of the diagonal process.

LEMMA 4. *If $\mathcal{D}$ is a diagonal process relative to the maximal abelian algebra $\mathcal{A}$, there is a *-representation, ϕ, of the algebra, $\mathcal{B}$, of all bounded operators which is an isomorphism on $\mathcal{A}$, and a projection E on the representation space such that*

$$\mathcal{D}(B) = \phi^{-1}[E(\phi(B))E]$$

for each B in $\mathcal{B}$.

Proof. Let $\{F_\alpha\}$ be a maximal orthogonal family of countably decomposable projections in $\mathcal{A}$, so that $\sum_\alpha F_\alpha = I$. Note that $\mathcal{D}_\alpha$, defined by $\mathcal{D}_\alpha(B) = \mathcal{D}(B)F_\alpha$, for B in the algebra, $\mathcal{B}_\alpha$, of bounded operators on $F_\alpha(\mathcal{H})$ ($\mathcal{H}$ the underlying Hilbert space of $\mathcal{B}$), is a diagonal process relative to the maximal abelian algebra $\mathcal{A}F_\alpha$ (on $F_\alpha(\mathcal{H})$). If we can find a representation ϕ_α of $\mathcal{B}_\alpha$ and a projection E_α with the properties described in the lemma (relative to $\mathcal{A}_\alpha$), then the direct sum, ϕ, of the representations and E, the sum of E_α, establish the result for $\mathcal{D}$.

We may assume that $\mathcal{A}$ is countably decomposable, so that there exists a (unit) separating vector, x, for $\mathcal{A}$. We define a state, ρ, of $\mathcal{B}$ by: $\rho(B) = \omega_x(\mathcal{D}(B))$. From Gelfand-Neumark [1] and Segal [6], ρ gives rise to a *-representation ϕ of $\mathcal{B}$ constructed as follows. The set of operators B in $\mathcal{B}$ such that $\rho(B^*B) = 0$ is a left ideal, $\mathcal{J}$. The quotient vector space $\mathcal{B}/\mathcal{J}$ has a positive definite inner product on it defined by

$$[B + \mathcal{J}, C + \mathcal{J}] = \rho(C^*B),$$

so that the completion, $\mathcal{H}_0$, of $\mathcal{B}/\mathcal{J}$ relative to this inner product is a Hilbert space. The mapping, $B + \mathcal{J} \to AB + \mathcal{J}$, on $\mathcal{B}/\mathcal{J}$ to $\mathcal{B}/\mathcal{J}$ extends to a bounded operator $\phi(A)$, for each A in $\mathcal{B}$ and ϕ is the *-representation in question. That ϕ is an isomorphism on $\mathcal{A}$ (with range $\mathfrak{A}_0$, let us say) is a

consequence of the definition of ρ. In fact, if $\phi(A)=0$, then

$$0=[A+\mathfrak{J},A+\mathfrak{J}]=\rho(A^*A)=\omega_x(\mathcal{D}(A^*A))=\|Ax\|^2,$$

for A in $\mathfrak{A}$, whence $A=0$. (Recall that x was chosen as a separating vector for $\mathfrak{A}$.)

Let E be the projection on the closure of $\{A+\mathfrak{J}: A\in\mathfrak{A}\}$. Our final assertion is that $\phi[\mathcal{D}(B)]=E\phi(B)E$, both operators restricted to $E(\mathfrak{H}_0)$. We have

$$\begin{aligned}[\phi[\mathcal{D}(B)](A+\mathfrak{J}),C+\mathfrak{J}]&=\rho(C^*\mathcal{D}(B)A)\\ &=\omega_x[C^*\mathcal{D}(B)A]=\omega_x[\mathcal{D}(C^*\mathcal{D}(B)A)]\end{aligned}$$

and

$$\begin{aligned}[E\phi(B)E(A+\mathfrak{J}),C+\mathfrak{J}]&=[\phi(B)(A+\mathfrak{J}),C+\mathfrak{J}]\\ &=\rho(C^*BA)=\omega_x[\mathcal{D}(C^*BA)]=\omega_x[C^*\mathcal{D}(B)A],\end{aligned}$$

with C and A in $\mathfrak{A}$. Thus, as operators on $E(\mathfrak{H}_0)$, $\phi[\mathcal{D}(B)]=E\phi(B)E$.

Remark 4. Relative to the scalar algebra, $\{\lambda I\}$ the identity mapping on $\mathfrak{B}$ is the unique proper diagonal process, and each state, ρ, of $\mathfrak{B}$ yields an improper diagonal process by means of the mapping $B\rightarrow\rho(B)I$.

Remark 5. If $\mathcal{D}$ is a diagonal process relative to $\mathfrak{A}_c$, then $\mathcal{D}(C)=0$ for each completely continuous operator, C. In fact, if ρ is a pure state of $\mathfrak{A}_c$, $\rho\circ\mathcal{D}$ is a state extension of ρ from $\mathfrak{A}_c$ to $\mathfrak{B}$ and so, the finite convex combinations of pure state extensions, ρ', of ρ to $\mathfrak{B}$ have $\rho\circ\mathcal{D}$ as a w^*-limit point. Now $\rho'(C)=0$ or else ρ' is a vector state, ω_x. But then ω_x is pure on $\mathfrak{A}_c$, so that x is a simultaneous eigenvector for $\mathfrak{A}_c$—a contradiction. Thus $\rho'(C)=0$, so that finite convex combinations of such ρ' annihilate C and $\rho\circ\mathcal{D}(C)=\rho[\mathcal{D}(C)]=0$. Hence $\mathcal{D}(C)=0$.

3. Uniqueness and non-uniqueness of diagonal processes. We consider $\mathfrak{A}_d$ first and show that there is a unique diagonal process relative to it. Let $\{x_k\}$ be an orthonormal basis for $\mathfrak{H}$, the Hilbert space upon which $\mathfrak{A}_d$ acts, relative to which each operator in $\mathfrak{A}_d$ is diagonal. Let us define $\mathcal{D}(B)$ for a bounded operator, B, to be the operator whose matrix representation relative to $\{x_k\}$ is the diagonal matrix with diagonal that of the matrix representation for B relative to $\{x_k\}$. Clearly, then, $\mathcal{D}$ is a diagonal process relative to $\mathfrak{A}_d$. With $x=\sum_k\alpha_kx_k$ and $\|B\|\leqq1$,

$$|(\mathcal{D}(B)x,x)|\leqq\sum|\alpha_k|^2|(\mathcal{D}(B)x_k,x_k)|=\sum|\alpha_k|^2|(Bx_k,x_k)|,$$

and for suitably large N, $\sum_{k \geqq N} |\alpha_k|^2 |(Bx_k, x_k)| < \epsilon/2$. Thus, with

$$|(Bx_k, x_k)| < \epsilon/2 \, \| x \|, \qquad k = 1, \cdots, N,$$

we have $|(\mathcal{D}(B)x, x)| \leqq \sum |\alpha_k|^2 |(Bx_k, x_k)| < \epsilon$, so that $\mathcal{D}$ is a continuous mapping at 0 on the unit ball of the algebra, $\mathcal{B}$, of all bounded operators in the weak operator topology into $\mathcal{B}$ in this topology. Since $\mathcal{B}$ is a topological linear space in the weak operator topoloygy and $\mathcal{D}$ is linear, $\mathcal{D}$ is continuous on the unit ball of $\mathcal{B}$ in this topology. From Lemma 2, it follows that $\mathcal{D}$ is the unique proper diagonal process relative to $\mathcal{A}_d$. By other considerations, we show that $\mathcal{D}$ is the *unique diagonal process* relative to $\mathcal{A}_d$.

THEOREM 1. *The unique diagonal process relative to $\mathcal{A}_d$ is $\mathcal{D}$.*

Proof. If $\mathcal{D}'$ is a diagonal process distinct from $\mathcal{D}$, then $\mathcal{D}'(B) \neq \mathcal{D}(B)$ for some B in $\mathcal{B}$; so that $(\mathcal{D}'(B)x_k, x_k) \neq (\mathcal{D}(B)x_k, x_k)$, for some k—whence $\omega_{x_k} \circ \mathcal{D}' \neq \omega_{x_k} \circ \mathcal{D}$. But ω_{x_k} is a vector pure state of $\mathcal{A}_d$ and has a unique state extension to $\mathcal{B}$. Thus $\mathcal{D}$ is the unique diagonal process relative to $\mathcal{A}_d$.

Of course, this does not establish that the pure states of $\mathcal{A}_d$ which are not vector states (the points of the β-compactification of the integers other than integer points) have unique state extensions to $\mathcal{B}$.

THEOREM 2. *There is more than one proper diagonal process relative to $\mathcal{A}_c$; pure state extension is not unique relative to $\mathcal{A}_c$.*

Proof. If we represent our Hilbert space, $\mathcal{H}$ as $L_2(0,1)$ under Lebesgue measure and $\mathcal{A}_c$ as the multiplication algebra of this measure space, then the set of projections $\{E_{km} : m = 1, 2, \cdots, k = 1, \cdots, m\}$ corresponding to multiplication by the characteristic function of the closed intervals $[(k-1)/m, k/m]$ generate $\mathcal{A}_c$. Now $I = \sum_{k=1}^{m} E_{km}$, so that $B^{|E_{1m}|\cdots|E_{mm}} = \sum_{k=1}^{m} E_{km}BE_{km}$, and $B^{|E_{1m}|\cdots|E_{mm}|E_{1mn}|\cdots|E_{mnmn}} = \sum_{k=1}^{mn} E_{kmn}BE_{kmn}$. From Lemma 2, if there is a unique diagonal process $\mathcal{D}$ of the form $\mathcal{D}_p$, p in the β-compactification of the integers but not an integer, in particular, if there is a unique proper diagonal process, then $\mathcal{D}(B)$ is the weak limit (with respect to j) of $\sum_{k=1}^{m} E_{km}BE_{km}$, where $m = 2^j$. In fact, if this is not the case for some B, then $\mathcal{D}_p(B) \neq \mathcal{D}_{p'}(B)$ for some points p and p' in $\beta(\mathcal{I}) - \mathcal{I}$. We shall exhibit such a B.

The functions, f_n, defined by $f_n(x) = e^{2\pi inx}$, for $n = 0, \pm 1, \cdots$, form an orthonormal basis for $\mathcal{H}$. As B, we shall take the projection, G, on the subspace spanned by certain of these elements $\{f_{n_j} : j = 1, 2, \cdots\}$ (to be

specified later). We have

$$(E_{km}GE_{km}(1),1)=\sum_{j=1}^{\infty}|(f_{n_j},E_{km}(1))|^2=\sum_{j=1}^{\infty}|\int_{(k-1)/m}^{k/m}e^{2\pi i n_j x}\,dx\,|^2$$

$$=\sum_{j=1}^{\infty}|(1/2\pi i n_j)[e^{2\pi i n_j k/m}-e^{2\pi i n_j(k-1)/m}]|^2$$

$$=\sum_{j=1}^{\infty}(1/4\pi^2 n_j^2)\,|\,e^{2\pi i n_j/m}-1\,|^2,$$

whence

$$((\sum_{k=1}^{m}E_{km}GE_{km})(1),1)=\sum_{j=1}^{\infty}(m/4\pi^2 n_j^2)\,|\,e^{2\pi i n_j/m}-1\,|^2$$

$$=\sum_{j=1}^{\infty}(m/\pi^2 n_j^2)\sin^2(\pi n_j/m).$$

We show that, for a suitable choice of $n_1, n_2, \cdots$,

$$(1)\qquad (m/\pi^2)\sum_{j=1}^{\infty}(1/n_j^2)\sin^2(\pi n_j/m)$$

does not tend to a limit as m $(=2^r)$ tends to ∞. For our set, $\{n_j\}$, choose all integers in the closed intervals $[2^{2^k-2},2^{2^k-1}]$, $k=1,2,\cdots$ (so that $n_1=1$, $n_2=2, n_3=4, n_4=5,\cdots$).

Note that (1) may be rewritten as

$$(1/\pi)\sum_{j=1}^{\infty}(\pi n_j/m)^{-2}\,[\sin^2(\pi n_j/m)]\,(\pi/m)\,(=a_m)$$

which is the integral over $[0,\infty]$ of the step function, s_m, defined as $(\pi n_j/m)^{-2}[\sin^2(\pi n_j/m)]\cdot(1/\pi)$ on the interval $[\pi(n_j-1)/m,\pi n_j/m]$, $j=1,2,\cdots$, and 0 elsewhere, and that, with $m\equiv 0$ (4),

$$(1/\pi)\sum_{m/4<n_j\leqq m/2}(\pi n_j/m)^{-2}\,[\sin^2(\pi n_j/m)]\cdot(\pi/m)$$

$$=\int_{\pi/4}^{\pi/2}s_m(x)\,dx\;(=b_m).$$

Now, with $m_k=2^{2^k}$, s_{m_k} is a Riemann approximating step function to $\pi^{-1}x^{-2}\sin^2 x$, on the interval $[\pi/4,\pi/2]$. Thus, if a_{2^m} tends to a limit as m tends to ∞, so does a_{m_k} as k tends to ∞, and

$$\lim_m a_{2^m}=\lim_k a_{m_k}\geqq\lim_k b_{m_k}=\pi^{-1}\int_{\pi/4}^{\pi/2}x^{-2}\sin^2 x\,dx>\pi^{-2}.$$

(For the last inequality, note that the derivative, $2x^{-3}\sin x\;(x\cos x-\sin x)$, of $x^{-2}\sin^2 x$ is negative on $[\pi/4,\pi/2]$, so that $x^{-2}\sin^2 x>4\pi^{-2}$ on $[\pi/4,\pi/2)$.)

On the other hand, there are no terms n_j in $(2^{2^k-1}, 2^{2^{k+1}-2})$. Thus, with $r_k = 2^{2^{k+1}-2}$, s_{r_k} is 0 on $[\pi 2^{2^k-1}(2^{2^{k+1}-2})^{-1}, \pi k/(k+1)]$, whose left end point tends to 0 as k tends to ∞. Since each s_m is bounded (e.g. by 1) on $[0,\pi]$, we have $\lim_{k\to\infty}\int_0^\pi s_{r_k}(x)\,dx = 0$. Thus, if $\lim_m a_{2^m}$ exists, then

$$\pi^{-2} < \lim_k a_{m_k} = \lim_k a_{r_k} = \lim_k \int_0^\infty s_{r_k}(x)\,dx$$
$$= \lim_k \int_\pi^\infty s_{r_k}(x)\,dx \leqq \lim_k (1/\pi) \int_\pi^\infty x^{-2}\,dx = \pi^{-2},$$

a contradiction. (Note that $s_m(x) \leqq \pi^{-1}x^{-2}$, for x in $[0,\infty)$.) Thus $\lim_k a_{2^k}$ does not exist, and G does not have a unique diagonal part relative to $\mathcal{A}_c$ (G is the projection on the space spanned by $e^{2\pi i n_j x}$, where $\{n_j\}$ is as described). From our earlier discussion, there are pure states of $\mathcal{A}_c$ which do not have unique state (and pure state) extensions to all bounded operators (in fact, which have distinct values on G).

4. The pure states. We have noted that non-uniqueness of diagonal processes implies non-uniqueness of pure state extension and that uniqueness of the diagonal process does not lead to uniqueness of pure state extension. The problem of uniqueness of pure state extension (and even that of diagonal processes) may be raised in more refined form. Given a maximal abelian self-adjoint algebra $\mathcal{A}$; for which operators, B, is it the case that all extensions of the same pure state of $\mathcal{A}$ coincide on B?

LEMMA 5. *If $\mathcal{A}$ is a maximal abelian algebra then there exists a sequence of projections $\{E_n\}$ in $\mathcal{A}$ such that $B^{|E_1|\cdots|E_n}$ converges to an operator of $\mathcal{A}$ in the uniform topology if and only if $\rho_1(B) = \rho_2(B)$ for each pair of states, ρ_1, ρ_2, of all bounded operators such that $\rho_1 \mid \mathcal{A} = \rho_2 \mid \mathcal{A}$ is a pure state of $\mathcal{A}$.*

Proof. Suppose that a sequence such as $\{E_n\}$ exists, for the operator B. Then, with ρ_1 and ρ_2 states of all bounded operators whose restrictions to $\mathcal{A}$ are pure and equal, $\rho_1(B^{|E}) = \rho_1(B)$ and $\rho_2(B^{|E}) = \rho_2(B)$ for each projection, E, in $\mathcal{A}$. In fact, $\rho_1(E)$ is 0 or 1, since E is a projection in $\mathcal{A}$ and ρ_1 is pure on $\mathcal{A}$, while $\rho_1(B^{|E}) = \rho_1(E)\rho_1(B)\rho_1(E) + \rho_1(I-E)\rho_1(B)\rho_1(I-E) = \rho_1(B)$ (cf. Lemma 2, second paragraph of the proof). Thus,

$$\rho_1(B^{|E_1|\cdots|E_n}) = \rho_1(B) \text{ and } \lim \rho_1(B^{|E_1|\cdots|E_n}) = \rho_1(A) = \rho_1(B),$$

where $B^{|E_1|\cdots|E_n}$ tends uniformly to the operator A in $\mathcal{A}$ (recall that states of C^*-algebras are continuous in the uniform topology). Similary, $\rho_2(B) = \rho_2(A)$ $(= \rho_1(A) = \rho_1(B))$, so that $\rho_1(B) = \rho_2(B)$.

Suppose now that all extensions of each given pure state of $\mathcal{A}$ coincide on B. Clearly then each diagonal process relative to $\mathcal{A}$ has the same value, A, at B. We shall find $\{E_n\}$ such that $B^{|E_1|\cdots|E_n}$ tends uniformly to A, or equivalently, that $(B-A)^{|E_1|\cdots|E_n}=B^{|E_1|\cdots|E_n}-A$ tends uniformly to 0. Of course, extensions of a given pure state of $\mathcal{A}$ coincide on $B-A$, and have value 0 (since $B-A$ has diagonal 0 under each diagonal process relative to $\mathcal{A}$). We may assume, therefore, that each extension of a pure state of $\mathcal{A}$ has value 0 on B, and that B is self adjoint.

Now $\mathcal{A}$ is *-isomorphic with $C(X)$, where X is extremely disconnected—each point, x_0, of X corresponds to a pure state, ρ_{x_0}, of $\mathcal{A}$ (and conversely). Since each state extension of ρ_{x_0} has the value 0 on B, we have

$$0=\inf\{\rho_{x_0}(A): A \text{ in } \mathcal{A}, A\geqq B\}=\sup\{\rho_{x_0}(A): A \text{ in } \mathcal{A}, B\geqq A\}.$$

Thus, we can choose operators A_{x_0} and A^{x_0} in $\mathcal{A}$ such that $A^{x_0}\geqq B\geqq A_{x_0}$ and $1/n>\bar{A}^{x_0}(x_0)\geqq 0\geqq \bar{A}_{x_0}(x_0)>-1/n$, where $\bar{A}$ is the function in $C(X)$ corresponding to an operator, A, in $\mathcal{A}$. It follows that there is a closed-open set, containing x_0, whose characteristic function corresponds to a projection E_{n,x_0} in $\mathcal{A}$, on which $\bar{A}^{x_0}$ is less than $1/n$ and $\bar{A}_{x_0}$ is greater than $-1/n$. Then

$$\begin{aligned}(1/n)E_{n,x_0}&\geqq E_{n,x_0}A^{x_0}\\ &=E_{n,x_0}A^{x_0}E_{n,x_0}\geqq E_{n,x_0}BE_{n,x_0}\geqq E_{n,x_0}A_{x_0}\geqq(-1/n)E_{n,x_0},\end{aligned}$$

so that $\|E_{n,x_0}BE_{n,x_0}\|\leqq 1/n$. Since X is compact, the closed-open sets corresponding to E_{n,x_0}, for each x_0, cover X and have a finite subcovering. Denote the corresponding projections by $E_{n1},\cdots,E_{nk_n}$. We may replace this set of projections (using intersections and relative complements) by an orthogonal set of projections in $\mathcal{A}$ each of which is contained in some E_{nj} and such that each E_{nj} is the sum of projections in the new finite set. If E is one of the new projections, contained, say, in E_{n1}, then

$$\|EBE\|=\|EE_{n1}BE_{n1}E\|\leqq\|E\|^2\|E_{n1}BE_n\|\leqq 1/n.$$

We may assume that $E_{n1},\cdots,E_{nk_n}$ are orthogonal, so that their sum is I (their corresponding closed-open sets cover X). Thus

$$\|B^{|E_{n1}|\cdots|E_{nk_n}}\|=\|\sum_{j=1}^{k_n}E_{nj}BE_{nj}\|\leqq 1/n.$$

The sequence $E_{11},E_{12},\cdots,E_{1k_1},E_{21},\cdots,E_{2k_2},\cdots$ will serve as the desired sequence, $\{E_n\}$.

Remark 6. If the state extensions to $\mathcal{B}$ of a state of a maximal abelian

algebra, $\mathfrak{A}$, coincide on each operator of a certain set, $\mathcal{S}$, (i.e. the restriction of these extensions to $\mathcal{S}$ defines a single-valued function on $\mathcal{S}$) then the same is true for the uniform closure of the self-adjoint linear space generated by $\mathcal{S}$ and for the set of those operators, T, for which there is a family, $\{E_n\}$ of projections in $\mathfrak{A}$ such that $T^{|E_1|\cdots|E_n}$ has a uniform limit in $\mathcal{S}$.

THEOREM 3. *All state extensions to* $\mathfrak{B}$ *of a pure state of* $\mathfrak{A}_d$ *coincide on each permutation matrix* (i.e. *each linear operator which permutes the eigenvectors of* $\mathfrak{A}_d$).

Proof. Let $\{x_n\}_{n=1,2,\cdots}$ be a basis of eigenvectors for $\mathfrak{A}_d$, and let $Tx_n = x_{\alpha(n)}$, where α is a permutation of $\mathfrak{J}$. Then, the matrix of T relative to $\{x_n\}$ has 1 at each entry $\alpha(n)$, n and zeros at the other entries (it is a permutation matrix). Let $\mathfrak{J}_1$ be the fixed points of α. We shall define three other sets of integers, $\mathfrak{J}_2$, $\mathfrak{J}_3$, and $\mathfrak{J}_4$. Assign to $\mathfrak{J}_2$ the first element of $\mathfrak{J}$ not in $\mathfrak{J}_1$ and suppose that each element of $\mathfrak{J}$ less than n has been assigned to one of $\mathfrak{J}_1, \cdots, \mathfrak{J}_4$ in such a way that j and $\alpha(j)$ are not in the same set if they are distinct. Assign n to the first one of $\mathfrak{J}_2$, $\mathfrak{J}_3$, $\mathfrak{J}_4$ which contains neither $\alpha(n)$ nor $\alpha^{-1}(n)$, unless $\alpha(n) = n$, in which case, assign n to $\mathfrak{J}_1$. In this way, we construct four pairwise disjoint sets $\mathfrak{J}_1$, $\mathfrak{J}_2$, $\mathfrak{J}_3$, $\mathfrak{J}_4$ with union $\mathfrak{J}$ such that $\alpha(n)$ and n lie in no one of them, unless they are equal (in which case it lies in $\mathfrak{J}_1$). Let E_j, $j = 1, \cdots, 4$ be the projection (in $\mathfrak{A}_d$) on the subspace spanned by $\{x_k : k \text{ in } \mathfrak{J}_j\}$. From the construction, $E_1TE_1 = E_1$ and $E_jTE_j = 0$, $j = 2, 3, 4$, while $E_1, \cdots, E_4$ are mutually orthogonal and have sum I. Thus $T^{|E_1|E_2|E_3|E_4} = E_1$, which lies in $\mathfrak{A}_d$. An application of Lemma 5 completes the proof.

Combining Theorem 3 with Remark 6, we see that pure state extension from $\mathfrak{A}_d$ is unique to the algebra of linear combinations of permutation matrices (and to its uniform closure). We note that the proof of Theorem 3 applies to more general 0, 1 matrices (e.g. to those operators which annihilate some basis vectors and are one-one mappings of the others into the set of basis vectors).

5. Related questions. The results that we have obtained leave the question of uniqueness of extension of the singular pure states of $\mathfrak{A}_d$ open. We incline to the view that such extension is non-unique (although the diagonal process is unique—Theorem 1, and there is a large class of operators to which extension is unique—Theorem 3). Our considerations also raise the question of whether or not each pure state of $\mathfrak{B}$ is the extension of some pure state of some maximal abelian algebra. (This is true for the vector

states of $\mathfrak{B}$.) With regard to this last question, one can partially order the states of $\mathfrak{B}$ by comparing the sets of elements on which they give "definite information." (We say that the state, ω of $\mathfrak{B}$ is definite on a self-adjoint operator, A, when ω is pure on the C^*-algebra generated by A-equivalently, when $\omega(A^2) = \omega(A)^2$. The set of operators on which ω is definite is "the definite set" of ω.)

THEOREM 4. *A state of $\mathfrak{B}$ is pure if and only if its definite set is maximal (with respect to inclusion).*

Proof. If $\mathcal{K}$ is the left kernel of ω, then the definite set of ω is $\{\mathcal{K}_* + \lambda I\}$, where $\mathcal{K}_*$ is the set of self-adjoint operators in $\mathcal{K}$ and λ is a real number. (Note that the set of self-adjoint operators in a uniformly closed left ideal determines that ideal—in fact, the positive operators in the ideal determine it [7].) Indeed, if A is in $\mathcal{K}_*$, then $0 = \omega(A) = \omega(A)^2 = \omega(A^2)$ (by definition of $\mathcal{K}$), so that ω is definite on $\mathcal{K}_*$ and hence on $\{\mathcal{K}_* + \lambda I\}$. On the other hand, if ω is definite on B, then it is definite on $B - \omega(B)I$, so that $\omega([B-\omega(B)I]^2) = \omega(B-\omega(B)I)^2 = 0$ and $B - \omega(B)I$ is in $\mathcal{K}_*$, B is in $\{\mathcal{K}_* + \lambda I\}$. If $\mathcal{K}$ is not a maximal left ideal and $\mathcal{J}$ is a left ideal in $\mathfrak{B}$ containing $\mathcal{K}$ properly, choose A in $\mathcal{J}_*$ not in $\mathcal{K}$. If $A = B + \lambda I$, with B in $\mathcal{K}_*$, then $A - B = \lambda I$ is in $\mathcal{J}$, so that $\lambda = 0$, $A = B$ is in $\mathcal{K}$—a contradiction. Thus $\{\mathcal{J}_* + \lambda I\}$ contains $\{\mathcal{K}_* + \lambda I\}$ properly. It follows that the definite set of ω is maximal only if its left kernel is a maximal left ideal—which implies that ω is pure [2].

Suppose, now, that $\mathcal{K}$ is a maximal left ideal and that $\mathcal{J}$ is a left ideal such that $\{\mathcal{J}_* + \lambda I\}$ contains $\{\mathcal{K}_* + \lambda I\}$, the definite set of some pure state. We show that $\{\mathcal{J}_* + \lambda I\}$, the definite set of an arbitrary state, coincides with $\{\mathcal{K}_* + \lambda I\}$, in this case. Passing to a maximal left ideal containing $\mathcal{J}$, we may assume that $\mathcal{J}$ itself is maximal; so that $\mathcal{J}$ is the left kernel of a pure state, ρ, of $\mathfrak{B}$. If $\mathcal{J}$ annihilates a vector, y, then so must $\mathcal{K}$; for otherwise, $\mathcal{K}_*$ contains all self-adjoint completely continuous operators, and in particular, one which maps y onto a non-zero vector orthogonal to y-contradicting the fact that $\{\mathcal{J}_* + \lambda I\}$ has y as an eigenvector and contains $\{\mathcal{K}_* + \lambda I\}$. Thus $\mathcal{K}_*$ consists of all self-adjoint operators annihilating some vector, z, and has y as an eigenvector; so that z is a scalar multiple of y, $\mathcal{K}_*$ annihilates y, $\mathcal{K} = \mathcal{J}$, and $\{\mathcal{K}_* + \lambda I\} = \{\mathcal{J}_* + \lambda I\}$.

We may assume that $\mathcal{J}$ does not annihilate a vector and, so, contains $\mathcal{C}$, the ideal of completely continuous operators in $\mathfrak{B}$. Thus, ϕ, the irreducible representation of $\mathfrak{B}$ associated with ρ has $\mathcal{C}$ as kernel. If A is in $\mathcal{K}_*$ but not in $\mathcal{J}_*$, then $\rho(A^2) = \rho(A)^2 \neq 0$; so that $\rho(E) \neq 0$ for some spectral

projection, E, of A corresponding to an interval whose closure does not contain 0. In fact, from the Spectral Theorem, A is a uniform limit of finite linear combinations of such spectral projections, and if ρ annihilates each of them, then since ρ is uniformly continuous, $\rho(A)=0$. Since E corresponds to an interval whose closure does not contain 0, $I-E+AE$ has an inverse, B; so that $BEA=E[B(I-E+AE)]=E$ is in $\mathcal{K}_*$ and hence in $\{\mathcal{J}_*+\lambda I\}$. Moreover, ECE is in $\mathcal{K}_*$, hence in $\{\mathcal{J}_*+\lambda I\}$, for each self-adjoint C in $\mathcal{B}$. Now ϕ maps $\{\mathcal{J}_*+\lambda I\}$ into the set of self-adjoint operators which have x as an eigenvector, where x is a vector such that $\omega_x\phi=\rho$; so that the projection, $\phi(E)$, has x in its range (since $\rho(E)\neq 0$), and $\phi(E)\phi(C)\phi(E)x=\phi(E)\phi(C)x=ax$. Since ϕ is an irreducible representation of $\mathcal{B}$, $\phi(E)$ must be the one-dimensional projection whose range contains x. But ρ annihilates $\mathcal{C}$, and $\rho(E)\neq 0$. Thus E is infinite dimensional, and $E=F+I-F$, where F and $I-F$ are infinite dimensional; so that $\phi(E)=\phi(F)+\phi(I-F)$, with $\phi(F)$ and $\phi(I-F)$ non-zero orthogonal projections. (Recall that the kernel of ϕ is $\mathcal{C}$). Hence $\phi(E)$ cannot be one-dimensional, each A in $\mathcal{K}_*$ lies in $\mathcal{J}_*$, $\mathcal{K}$ is contained in $\mathcal{J}$ and $\mathcal{K}=\mathcal{J}$, by maximality, and $\{\mathcal{K}_*+\lambda I\}$ is a maximal definite set.

Presumably, the definite set of each pure state contains the set of self-adjoint elements of some (perhaps many) maximal abelian algebras. A general question of obvious interest is that of the classification of the irreducible representations of $\mathcal{B}$. We know from [3] that the separable ones are all unitarily equivalent (to the algebra of bounded operators on separable Hilbert space) and are associated with vector states. The vector states of $\mathcal{B}$ are unitarily equivalent. Is this the case for the singular pure states of $\mathcal{B}$? A clever counting argument of Kaplansky's shows that this is not so. In fact, each pure state of $\mathcal{A}_d$ has a pure state extension to $\mathcal{B}$, so that there are at least 2^C pure states of $\mathcal{B}$ (the pure state space of $\mathcal{A}_d$ is $\beta(\mathcal{S})$ which has cardinality 2^C), while there are only C operators (as can easily be seen from the matrix representation relative to a countable orthonormal basis.) Each unitary equivalence class contains at most C states, so that there are 2^C inequivalent singular pure states.

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A GENERAL THEORY OF ALGEBRAIC GEOMETRY OVER DEDEKIND DOMAINS, III.*

Absolutely Irreducible Models, Simple Spots.

By Masayoshi Nagata.

In the present part of this sequence of papers, we want to study absolutely irreducible models and simple spots.

First, in Chapter 5, we study the extensions of ground rings and introduce the notion of absolutely irreducible models (§ 1). Then we introduce the notion of product models (§ 2) and fibre bundles (§ 3). And then we study the notion of absolutely normal spots (§ 4) and we introduce the notion of a point set attached to an absolutely irreducible models (§ 5). Furthermore, we define the notions of order of inseparability, cycles, divisors of functions, ideals of a model and line bundles (§§ 6-10).

Secondly, in Chapter 6, we first prove that the set of simple spots in a model forms a model (§ 1) and then we consider the Jacobian criterion of simplicity (§ 2) and we study the notion of absolutely simple spots (§ 3). In § 4 we prove that the set of absolutely normal spots in a model forms a model and then in § 5 we consider the notion of simple points. Finally, in § 6, we derive conditions for the simplicity of a tensor product of simple spots.

Results assumed to be known: Besides the results assumed to be known in previous parts of this sequence of papers and results in them, we assume that the results in [3] and [4] are known.

Chapter 5. Absolutely Irreducible Models.

1. **Extension of ground rings.** Let M be a model of a function field L over a ground ring I. Let I^* be a ground ring (not necessarily a ground ring of L) which contains I and let $\mathfrak{p}^*$ be a prime ideal of $L \otimes_I I^*$ such that $\mathfrak{p}^* \cap I = 0$. Let L^* be the field of quotients of $(L \otimes I^*)/\mathfrak{p}^*$. Then we can regard L and I^* as subrings of L^* in the natural way. (Then L^* is the field

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of quotients of $I^*[L]$). Let $A_1, \cdots, A_n$ be affine models of L such that $M = \cup_i A_i$ and let $\mathfrak{o}_1, \cdots, \mathfrak{o}_n$ be affine rings of $A_1, \cdots, A_n$ respectively. Set $\mathfrak{o}^*_i = I^*[\mathfrak{o}_i]$. Then $\mathfrak{o}^*_i$ is an affine ring of L^* over I^*. Let A^*_i be the affine model defined by $\mathfrak{o}^*_i$ and set $M^* = \cup_i A^*_i$. Then

THEOREM 1. 1) *M^* is a model of L^* over I^*,* 2) *every spot in M^* dominates a spot in M and* 3) *M^* is the set of spots which are rings of quotients of $I^*[Q]$, where Q runs over all spots in M, hence M^* does not depend on the choice of the affine models A_i.*

This model M^* is called *an extension* of M over I^* *in the weak sense.* When $\mathfrak{p}^* = 0$, we call M^* *the extension* of M over I^* and M^* is denoted by $M \otimes I^*$.

Proof. Let Q^* be an arbitrary spot in M^* and let $\mathfrak{q}^*$ be the maximal ideal of Q^*. Q^* is a ring of quotients of some $\mathfrak{o}^*_i$, hence Q^* contains $\mathfrak{o}_i$. Therefore Q^* dominates $Q = (\mathfrak{o}_i)_{(\mathfrak{q}^* \cap \mathfrak{o}_i)}$, which proves 2) and we see that Q^* is a ring of quotients of $I^*[Q]$. Assume that $Q^{**} \in M^*$ corresponds to Q^*. Let Q' be a spot in M which is dominated by Q^{**}. Since there exists a place which dominates both Q^* and Q^{**}, hence also Q and Q', we see that Q corresponds to Q'. Since M is a model, $Q = Q'$. Therefore Q^* and Q^{**} are rings of quotients of the same ring $I^*[Q]$. By Lemma 2.1.1, we see that $Q^* = Q^{**}$, which proves 1), because M^* is the union of a finite number of affine models. We saw already that any spot Q^* in M^* is a ring of quotients of $I^*[Q]$ with $Q \in M$. Conversely, if $Q \in M$, then $I^*[Q]$ is a ring of quotients of some $\mathfrak{o}^*_i$ and therefore any spot Q^* which is a ring of quotients of $I^*[Q]$ is in M^*, and 3) is proved.

Remark 1. Let n be the transcendence degree of I^* over I (n may be infinite). If I is not a field and if I^* is a field, then

$$\dim M > \dim M^* \geqq \dim M - n - 1;$$

if both I and I^* are fields or if both I and I^* are not fields, then

$$\dim M \geqq \dim M^* \geqq \dim M - n.$$

Remark 2. If $\mathfrak{p}^*$ is a prime divisor of zero of $L \otimes I^*$, then $\dim M^*$ is equal to $\dim M$ or $\dim M - 1$, where the last case occurs when and only when I is not a field and I^* is a field. (For the proof, see Lemma 3.1.5.)

With the same L, L^* and I^*, let P be a spot of L (over I). Then a spot P^* which is a ring of quotients of $I^*[P]$ and which dominates P is called *an extension* of P over I^* *in the weak sense.* When $\mathfrak{p}^*$ is zero, P^* is called

an extension of P over I^*. By this definition, the extension of M over I^* is the set of extensions of spots in M over I^*.

In Chapter 2, we defined the notion of domination of models, which can be defined for models having different ground rings. If a model M^* dominates a model M, there exists a mapping f from M^* into M such that $f(P^*) = P$ $(P^* \in M^*, P \in M)$ if and only if P^* dominates P. This mapping f is called *the projection* or *the geometric projection* from M^* into M, which will be denoted by one of $\mathrm{proj}^{M^*}{}_M$, proj_M, proj.

Theorem 1 shows that from any extension of M even in weak sense the projection into M is well defined.

Assume that the extension $M \otimes I^*$ is defined. Then for a spot P in M, the set of spots P^* in $M \otimes I^*$ such that i) $\mathrm{proj}\, P^* = P$ and ii) $\mathrm{rank}\, P = \mathrm{rank}$ P^* is the set of rings of quotients of the local tensor product $P \times_I I^*$ with respect to its maximal ideals; such a P^* is called *a component* of $P \times I^*$. It will be easy to see that 1) if $Q^* \in M \otimes I^*$ and if $\mathrm{proj}\, Q^* = P$, then Q^* is a specialization of some component of $P \times I^*$ and 2) $P \times I^*$ is not defined if and only if there exists no ground place of I^* which dominates that ground place of I dominated by P.

We say that a model M over a ground ring I is *absolutely irreducible* if for any ground ring I^* containing I, the extension $M \otimes I^*$ is well defined, or equivalently, if the function field of M is a regular extension of I (by Theorem 3.3).

Remark. Let M be a normal model of a function field L over a ground ring I. Then every spot $P \in M$ contains the integral closure I' of I in L. Hence we may regard M as a model over I'. In this case, if L is separably generated over I, then L is a regular extension of I' (see Chapter 3, § 4) and M is an absolutely irreducible model over I'; if $I \neq I'$, then M is not an absolutely irreducible model over I.

LEMMA 1. *Let $\mathfrak{o}$ and $\mathfrak{o}'$ be rings containing a field k. If an element a of $\mathfrak{o}$ is not a zero divisor in $\mathfrak{o}$ then a is not a zero divisor in $\mathfrak{o} \otimes_k \mathfrak{o}'$.*

Proof. Let $\{v_i\}$ be a linearly independent k-base of $\mathfrak{o}'$. Assume that $ab = 0$ $(b \in \mathfrak{o} \otimes \mathfrak{o}')$ and $b = \sum a_i v_i$ $(a_i \in \mathfrak{o})$. Then $ab = 0$ means $\sum a a_i v_i = 0$, hence $a a_i = 0$ for all i. Therefore $a_i = 0$ and $b = 0$.

LEMMA 2. *Let $\mathfrak{o}$ and $\mathfrak{o}'$ be Noetherian rings containing a field k. Then any prime divisor of zero of $\mathfrak{o} \otimes_k \mathfrak{o}'$ is a minimal prime divisor of the ideal generated by some $\mathfrak{p}$ and $\mathfrak{p}'$ with $\mathfrak{p}$ and $\mathfrak{p}'$ prime divisors of zero in $\mathfrak{o}$ and $\mathfrak{o}'$ respectively.*

Proof. By the existence of a linearly independent k-base of $\mathfrak{o}'$, we see easily that if $\mathfrak{a}$ and $\mathfrak{b}$ are ideals of $\mathfrak{o}$, then

$$(\mathfrak{a}\cap\mathfrak{b})(\mathfrak{o}\otimes\mathfrak{o}')=\mathfrak{a}(\mathfrak{o}\otimes\mathfrak{o}')\cap\mathfrak{b}(\mathfrak{o}\otimes\mathfrak{o}').$$

Therefore we may assume that the zero ideal of $\mathfrak{o}$ is primary. Similarly, we assume also that the zero ideal of $\mathfrak{o}'$ is primary. Now, by Lemma 1, we may assume that the total quotient rings of $\mathfrak{o}$ and $\mathfrak{o}'$ coincide with $\mathfrak{o}$ and $\mathfrak{o}'$ respectively. Thus, we assume that $\mathfrak{o}$ and $\mathfrak{o}'$ are local rings of rank zero. If $\mathfrak{o}$ and $\mathfrak{o}'$ are fields, then the proof is easy (see the proof of Lemma 3. 1. 5). We shall use induction on length $\mathfrak{o}$ + length $\mathfrak{o}'$. Assume, for instance, that length $\mathfrak{o}>1$. Let b be an element of $\mathfrak{o}$ such that $0:b\mathfrak{o}=\mathfrak{p}$ with $\mathfrak{p}$ the maximal ideal of $\mathfrak{o}$. Let $\mathfrak{p}'$ be the maximal ideal of $\mathfrak{o}'$ and let $\mathfrak{p}^*_1,\cdots,\mathfrak{p}^*_n$ be all of the (minimal) prime divisors of the ideal generated by $\mathfrak{p}$ and $\mathfrak{p}'$. We have only to prove that if an element a of $\mathfrak{o}\otimes\mathfrak{o}'$ is not in any of the $\mathfrak{p}^*_i$, then a is not a zero divisor. Assume that $ac=0$ with $c\in\mathfrak{o}\otimes\mathfrak{o}'$. By the induction assumption, a is not a zero divisor modulo $b(\mathfrak{o}\otimes\mathfrak{o}')$, hence $c\in b(\mathfrak{o}\otimes\mathfrak{o}')$. Let c' be an element of $\mathfrak{o}\otimes\mathfrak{o}'$ such that $c=bc'$. Then $ac'\in 0:b(\mathfrak{o}\otimes\mathfrak{o}')$. By the existence of k-bases, we see easily that $0:b(\mathfrak{o}\otimes\mathfrak{o}')=\mathfrak{p}(\mathfrak{o}\otimes\mathfrak{o}')$. Again by the induction assumption, a is not a zero divisor modulo $\mathfrak{p}(\mathfrak{o}\otimes\mathfrak{o}')$. Therefore we have $c'\in\mathfrak{p}(\mathfrak{o}\otimes\mathfrak{o}')$; hence $c=bc'=0$. Therefore a is not a zero divisor, which proves our assertion.

LEMMA 3. *Let $\mathfrak{p}$ be a prime ideal of a spot P over a ground ring I and let I^* be a ground ring containing I. If $\mathfrak{p}^*$ is a prime divisor of the ideal $\mathfrak{p}(P\otimes_I I^*)$, then there exists a (minimal) prime divisor $\mathfrak{m}^*$ of the ideal of $P\otimes I^*$ generated by the maximal ideal $\mathfrak{m}$ of P such that $\mathfrak{p}^*\subseteq\mathfrak{m}^*$.*

Proof. Since $P\otimes I^*/\mathfrak{p}(P\otimes I^*)=(P/\mathfrak{p})\otimes(I^*/(\mathfrak{p}\cap I)I^*)$, we may assume that $\mathfrak{p}=0$. By Lemma 3. 1. 5, $\mathfrak{p}^*$ is a minimal prime divisor of zero. Assume that we know that $\mathfrak{p}^*+\mathfrak{m}(P\otimes I^*)\neq P\otimes I^*$. Since P is a ring of quotients of an affine ring over I, say $I[x_1,\cdots,x_n]$, $P\otimes I^*$ is a ring of quotients of $I^*[x_1,\cdots,x_n]$ $(=I[x_1,\cdots,x_n]\otimes I^*)$. $\mathfrak{p}^*\cap I^*[x_1,\cdots,x_n]$ is a minimal prime divisor of zero of $I^*[x_1,\cdots,x_n]$. Therefore the transcendence degree of $P\otimes I^*/\mathfrak{p}^*$ over I^* is that of P over I. Therefore, if $\mathfrak{q}^*$ is a prime ideal of $P\otimes I^*$ containing $\mathfrak{p}^*$ then rank $(\mathfrak{q}^*/\mathfrak{p}^*)=\operatorname{rank}\mathfrak{q}^*$. Now, let $y_1,\cdots,y_m$ be a system of parameters of P and let $\mathfrak{m}^*$ be a minimal prime divisor of $\mathfrak{p}^*+\sum y_i(P\otimes I^*)$. Then we have, by the above observation, $\operatorname{rank}\mathfrak{m}^*\leqq m$. Since $\mathfrak{m}^*$ contains $\mathfrak{m}$, and since every prime ideal of $P\otimes I^*$ containing $\mathfrak{m}$ is of rank not less than m, we see that $\mathfrak{m}^*$ is a minimal prime divisor of $\mathfrak{m}(P\otimes I^*)$. Therefore it is sufficient to show that $\mathfrak{p}^*+\mathfrak{m}(P\otimes I^*)$

$\neq P\otimes I^*$, assuming that $\mathfrak{m}(P\otimes I^*)\neq P\otimes I^*$. For the purpose, we may assume that I^* is a ring of quotients of an affine ring over I. By Lemma 1, we may assume that I and I^* are ground places and I is dominated by P and I^*. If $x^*_1,\cdots,x^*_r$ are algebraically independent elements over I such that $I(x^*_1,\cdots,x^*_r)$ is dominated by I^*, then the zero ideal and the ideal $\mathfrak{m}(P\otimes I(x^*_1,\cdots,x^*_r))$ of $P\otimes I(x^*_1,\cdots,x^*_r)$ are obviously prime. Therefore, considering $(P\otimes I(x^*_1,\cdots,x^*_r))_{(\mathfrak{m})}$ and $I(x^*_1,\cdots,x^*_r)$ instead of P and I respectively, we may assume that I^* is a ring of quotients of an integral extension I' of I. (Therefore, if I is a field, then the proof is easy.) We first consider the case where $\operatorname{rank} P=1$. Then $P\otimes I^*$ is a semi-local ring and every maximal ideal of $P\otimes I^*$ is of rank 1, as is easily seen. Therefore we see the proof of this case. Then the general case is proved by induction on rank P.

THEOREM 2. *Let M be a model of a function field L over a ground ring I and let I^* be a ground ring containing I.*

1) *If M is an absolutely irreducible model, then $M\otimes I^*$ (is well defined and) is an absolutely irreducible model over I^*.*

2) *Assume that $M\otimes I^*$ is well defined and let P be a spot in M. If an element f of $L\otimes I^*$ is in every component of $P\times I^*$, then f is in $P\otimes I^*$, provided that $P\times I^*$ is defined.*

Proof. 1) follows from Corollary 1 to Theorem 3.3. As for 2), we may assume that I is a ground place dominated by P. f is expressed in the form $\sum_1^n c_iw_i$ with $c_1,\cdots,c_n\in I^*$ which are linearly independent over I and with $w_i\in L$.

i) The case where I is a field: Let a be an element of P such that $aw_i\in P$ for any i $(a\neq 0)$. By Lemmas 2 and 3, we see that

$$\cap\,(aP_i\cap(P\otimes I^*))=a(P\otimes I^*),$$

where P_i runs over all components of $P\times I^*$. Since $f\in P_i$, $af\in aP_i$. Therefore $af\in a(P\otimes I^*)$. Since a is not a zero divisor by Lemma 1, we have $f\in P\otimes I^*$.

ii) The case where I is not a field: Let p be a prime element of I. By case i) and by Lemma 3, the assertion is true for $P_{\mathfrak{q}}$ if $\mathfrak{q}$ is a prime ideal of P which does not contain p, i.e., $f\in\cap_{\mathfrak{q}}P_{\mathfrak{q}}\otimes I^*$ ($\mathfrak{q}$ runs over all prime ideals of P which do not contain p). Therefore there exists a power p^r of p such that $p^rf\in P\otimes I^*$. Choose r the smallest possible one. If $r=0$, then

$f \in P \otimes I^*$. Assume the contrary. If $r > 1$, then considering $p^{r-1}f$, we may assume that $r = 1$. Then, since $P \otimes I^*/p(P \otimes I^*) = (P/pP) \otimes_{I/pI} (I^*/pI^*)$ and since I/pI is a field, we have $f \in P \otimes I^*$ as in the case i), which gives the proof for ii).

THEOREM 3. *If M is an absolutely irreducible model over a ground ring I, then for all but a finite number of prime ideals $\mathfrak{p}$ of I,* 1) *there exists only one general spot $P(\mathfrak{p})$ over $\mathfrak{p}$ in M,* 2) *$P(\mathfrak{p})$ is an unramified simple spot of rank* 1 *and* 3) *the induced model $\phi_{P(\mathfrak{p})}(M)$ is an absolutely irreducible model over $I/\mathfrak{p}$.*

Proof. When M is an affine model, the assertion is immediate from Theorem 3.6. As for the general case, let A be an affine model contained in M and let $\mathfrak{p}$ be a prime ideal of I. If one of the three conditions in our theorem is not true of $\mathfrak{p}$, then either there exists a general spot $P(\mathfrak{p})$ contained in $M - A$ or one of the conditions is not true with respect to A. Since $P(\mathfrak{p})$ is of rank 1, the first case is true only for a finite number of $\mathfrak{p}$, while by the case of affine models, the same is true for the second case. Therefore Theorem 3 is proved.

2. Product models. Let M and M' be models of function fields L and L' respectively over the same ground ring I. Assume that $L \otimes_I L'$ is an integral domain. (By Theorem 3.3, if one of M and M' is absolutely irreducible, then $L \otimes L'$ is an integral domain.) Let L^* be the field of quotients of $L \otimes L'$ and we regard L and L' as subfields of L^* in the natural way ($L^* = L(L')$). In this case, the join $J(M, M')$ of M and M' is called *the product model* of M and M' and will be denoted by $M \otimes M'$. When I is a field, $\dim (M \otimes M') = \dim M + \dim M'$; when I is not a field, $\dim M \otimes M' = \dim M + \dim M' - 1$.

PROPOSITION 1. 1) *If M and M' are affine models, then $M \otimes M'$ is also an affine model.* 2) *If M and M' are complete models, then $M \otimes M'$ is also a complete model.* 3) *If M and M' are projective models, then $M \otimes M'$ is also a projective model.*

The proof is immediate from the results in Chapter 2, § 4.

PROPOSITION 2. *When L and L' are imbedded in some other function field, then the join $J(M, M')$ in this case can be regarded as an induced model of $M \otimes M'$.*

Proof. Set $\mathfrak{S} = L[L']$ in the new imbedding. Then $\mathfrak{S}$ is a homomorphic

image of $L \otimes L'$. Let $\mathfrak{q}^*$ be the kernel of the homomorphism. On the other hand, let $\mathfrak{o}$ and $\mathfrak{o}'$ be affine rings whose affine models are contained in M and M' respectively. Set $\mathfrak{q} = \mathfrak{q}^* \cap (\mathfrak{o} \otimes \mathfrak{o}')$. Then $Q^* = (\mathfrak{o} \otimes \mathfrak{o}')_{\mathfrak{q}}$ is a spot in $M \otimes M'$. Since $L \otimes L'$ is a ring of quotients of $\mathfrak{o} \otimes \mathfrak{o}'$, we have $Q^* = (L \otimes L')_{\mathfrak{q}^*}$. Therefore we see easily that $J(M, M')$ is the induced model of $M \otimes M'$ defined by the spot Q^*.

PROPOSITION 3. *Let L, L' and L^* be the same as above ($L^* = L(L')$). Then a spot P of L corresponds to a spot P' of L' if and only if P and P' dominate the same ground place (of I).*

Proof. The only if part is obvious. Assume that P and P' dominate the same ground place I'. Let $\mathfrak{m}$ and $\mathfrak{m}'$ be maximal ideals of P and P' respectively and let I'' be the residue class field of I'. Furthermore, let $\mathfrak{a}$ be the ideal of $P[P']$ $(= P \otimes P')$ generated by $\mathfrak{m}$ and $\mathfrak{m}'$. Then $P[P']/\mathfrak{a} = (P/\mathfrak{m}) \otimes_{I''} (P'/\mathfrak{m}')$, which shows that $\mathfrak{a}$ does not contain the identity. Therefore P corresponds to P' by Theorem 2.1.

COROLLARY. *Let M and M' be models over a ground ring I. Assume that $M \otimes M'$ is defined and that any ground place of I is dominated by some spots in M'. Then the projection from $M \otimes M'$ in M is an onto mapping.*

THEOREM 4. *Let M and M' be models over the same ground ring I. If one of M and M' is absolutely irreducible, then $M \otimes M'$ is well defined. If both M and M' are absolutely irreducible, then $M \otimes M'$ is also absolutely irreducible.*

The proof is immediate from the results in Chapter 3, § 4.

3. Definition of fibre bundles. Let M and F be models of function fields K and L respectively over the same ground ring I. Assume that $M \otimes F$ is well defined and let L^* be the function field of $M \otimes F$.

Let $\sigma_1, \cdots, \sigma_n$ be automorphisms of L^* over K and assume that there are models $A_1, \cdots, A_n$ of K such that

1) $M = \cup_i A_i$ and 2) $\sigma_i^{-1}\sigma_j((A_i \cap A_j) \otimes F) = (A_i \cap A_j) \otimes F$ for any (i, j). Then

THEOREM 5. *$M^* = \cup_i \sigma_i(A_i \otimes F)$ is a model of L^* over I.*

We call this model M^* *a fibre bundle* with *base model* M and *fibre* F; the σ_i's are called the *transition mappings* of M^* (defined on A_i's).

Proof. Since $A_i \otimes F$ is a model of L^*, $\sigma_i(A_i \otimes F)$ is also a model of L^*.

Therefore M^* is the union of a finite number of affine models. Assume that two spots P^* and P^{**} in M^* correspond to each other. Take i and j such that $P^* \in \sigma_i(A_i \otimes F)$, $P^{**} \in \sigma_j(A_j \otimes F)$. Let P' and P'' be the spots in M which are dominated by $\sigma_i^{-1}(P^*)$ and $\sigma_j^{-1}(P^{**})$ respectively. Since σ_i and σ_j are automorphisms over K, P^* and P^{**} dominate P' and P'' respectively, which shows that P' corresponds to P'' and therefore $P' = P''$. Therefore P' is in $A_i \cap A_j$. Therefore $\sigma_i^{-1}(P^*)$ and $\sigma_j^{-1}(P^{**})$ are in $(A_i \cap A_j) \otimes F$. Thus we have $P^* \in \sigma_i((A_i \cap A_j) \otimes F)$, $P^{**} \in \sigma_j((A_i \cap A_j) \otimes F)$. By our assumption, $\sigma_i((A_i \cap A_j) \otimes F) = \sigma_j((A_i \cap A_j) \otimes F)$. Therefore P^* and P^{**} are in the same model, hence $P^* = P^{**}$. Therefore M^* is a model.

When M^{**} is another fibre bundle with base model M and fibre F, we say that M^{**} is *equivalent* to M^* if there exists an automorphism σ of L^* over K such that $M^{**} = \sigma M^*$.

We shall show here that the notion of fibre variety in the sense of Weil [7] corresponds to our notion of the fibre bundle. A fibre variety W, with base V, fibre F, structure group G, transition functions s_{ij} and the field k of definition, is defined in the following manner:

V and F are abstract varieties and G is an automorphism group on F (for the definition, see Weil [7]), defined over k. $\{U_i\}$ is a finite covering of V by open sets in the sense of the k-topology. For each pair (i, j), s_{ij} is a rational mapping over k of V into G and defined at all points of $U_i \cap U_j$. For any triple (h, i, j), $s_{hj} = s_{hi}s_{ij}$ in $U_h \cap U_i \cap U_j$. Forming the union $\bigcup_i (U_i \times F)$, we define in this union the equivalence relation, denoted by $\sim$, namely; $(x, z) \sim (x', z')$ $((x, z) \in U_i \times F, (x', z') \in U_j \times F)$ if and only if $x = x'$ $(\in U_i \cap U_j)$ and $z' = s_{ji}(x)z$. Then the set W of equivalence classes becomes an abstract variety, which is birationally equivalent to $V \times F$.

We shall show that the set $S(W)$ of specialization rings of points of W over k is a fibre bundle in our sense, with base model $S(V)$ and fibre $S(F)$, where $S(V)$ and $S(F)$ are the sets of specialization rings of points in V and F respectively, over k. We denote by $S(U_i)$ the set of specialization rings of points of U_i over k. Let P and Q be independent generic points of V and F respectively, and let $(x) = (x_1, \cdots, x_m)$, $(z) = (z_1, \cdots, z_n)$ be the coordinates of representatives of P and Q in some affine representatives V' and F' of V and F respectively. Then we may regard $k(x)$, $k(z)$ and $k(x, z)$ as function fields of V, F and $V \times F$ respectively. Since $s_{ij}(x)$ is rational over $k(x)$ and is an automorphism of F, $s_{ij}(x)z$ is also in F and its coordinates (z') are rationally expressed in term of $k(x, z)$ and therefore the mapping $(z) \rightarrow (z')$ defines an automorphism of $k(x, z)$ over $k(x)$; that is, $s_{ij}(P)$ induces an automorphism of $k(x, z)$ over $k(x)$, which will be denoted by σ_{ij}.

On the other hand, the equivalence relation $\sim$ can be formulated as follows: We regard $s_{ij}(P)$ as a birational correspondence between $U_j \times F$ and $U_i \times F$ which maps $P \times Q$ to $P \times Q'$, where Q' is the point of F which has the representative $s_{ij}(x)z$ in F'. Then a point $R \times S$ in $U_i \times F$ is equivalent to $R' \times S'$ in $U_j \times F$ if and only if they are corresponding points under $s_{ij}(P)$. Therefore, the automorphism σ_{ij} of $k(x,z)$ identifies the specialization rings of equivalent points of $U_i \times F$ and $U_j \times F$. This shows also that $\sigma_{ij}((S(U_i) \cap S(U_j)) \otimes S(F)) = (S(U_i) \cap S(U_j)) \otimes S(F)$. Since $s_{hj} = s_{hi}s_{ij}$, we have, fixing one h, say 1, $\sigma_{ij} = \sigma_{1i}^{-1}\sigma_{1j}$. Therefore, if we denote σ_{1j} by σ_j, then we see that $S(W)$ is a fibre bundle in our sense with base $S(V)$, fibre $S(F)$ and the transition mappings σ_i.

4. Absolutely normal spots. A spot P over a ground ring I is called *absolutely normal* if 1) P is a regular extension of I and 2) for any ground ring I^* containing I (and such that $P \times I^*$ is defined), every extension of P over I^* is normal. Observe that I^* may be restricted only to fields or valuation rings which dominate the ground place of I dominated by P and whose field of quotients is finitely generated over that of I.

A model M is called *absolutely normal* if every spot of M is absolutely normal.

A spot P over a ground ring I is called *weakly absolutely normal* if 1) P is a regular extension of I and 2) if I^* is a ground ring which is also a valuation ring (or a field) unramified over the ground place I' of I dominated by P (i.e., the maximal ideal of I' generates that of I^*), then every extension of P over I^* is normal.

If we consider only models over fields, then weak absolute normality coincides with absolute normality. Furthermore,

PROPOSITION 3. *Let P be a weakly absolutely normal spot over a ground ring I. If P contains the field of quotients of I, then P is absolutely normal.*

Proof. We may assume that I is a field. If P is not absolutely normal, then there exists a ground ring I^* containing I such that 1) some extension of P over I^* is not normal and 2) the field of quotients K of I^* is finitely generated over I. Let k be a finite purely inseparable extension of I such that $k(K)$ is separably generated over k. By the assumption, $P[k] = P \otimes k$ is a normal spot over k. Let I^{**} be the integral closure of I^* in $k(K)$. Since both $P[k]$ and I^{**} are separably generated over the field k and since they are normal rings, $P[k] \otimes_k I^{**}$ is a normal ring by Theorem 3.8. This

contradicts the assumption that there is an extension of P over I^* which is not normal; for, $P[k] \otimes_k I^{**} = P \otimes_I I^{**} = (P \otimes_I I^*) \otimes_{I^*} I^{**}$.

THEOREM 6. *Let P be a normal spot over a ground place dominated by P. If P is a regular extension of I and if I and the residue class field k of I are perfect, then P is weakly absolutely normal.* (*In this case, I is either a field or a ring of characteristic zero.*)

Proof. If I is a field, then the assertion follows by Theorem 3. 8. Assume that I is not a field. Let I^* be a ground place dominating I and unramified over I. Let x be a prime element of I. Set $\mathfrak{o}^* = P \otimes I^*$ and let $\mathfrak{p}^*$ be a prime divisor of $x\mathfrak{o}^*$. Since $\mathfrak{o}^*/x\mathfrak{o}^* = (P/xP) \otimes_{I/xI} I^*/xI^*$, $x\mathfrak{o}^*$ has no imbedded prime divisors, hence $\mathfrak{p}^*$ is a minimal prime divisor of $x\mathfrak{o}^*$. Set $\mathfrak{p} = \mathfrak{p}^* \cap P$. Then $\mathfrak{p}$ is a minimal prime divisor of xP. Since I^*/xI^* is a separably generated extension of I/xI, $P/\mathfrak{p} \otimes I^*/xI^*$ has no nilpotent elements. Therefore, if y is a prime element of $P_{\mathfrak{p}}$, then $\mathfrak{p}^*\mathfrak{o}^*_{\mathfrak{p}^*}$ is generated by y, which proves that $\mathfrak{o}^*_{\mathfrak{p}^*}$ is a normal ring. Therefore by Proposition 3. 9, the assertion follows.

COROLLARY 1. *Let P be a spot over a field k such that P is a regular extension of k. If there exists a field K containing k such that at least one extension of P over K is absolutely normal, then P is absolutely normal.*

Proof. Let k^* be the smallest perfect field containing k. We may assume that K contains k^*. Since there exists only one extension P^* of P over k^*, P^* must be normal, hence P^* is absolutely normal by Theorem 6, which shows that P is absolutely normal.

COROLLARY 2. *Let P be a spot over a field k such that P is a regular extension of k. Then P is absolutely normal if and only if $P[k']$ is normal for any finite purely inseparable extensions k' of k, or equivalently, for the smallest perfect field k' containing k.*

LEMMA 1. *Let P be a normal spot over a ground ring I and let x be a prime element of the ground place I' dominated by P. Assume that $x \neq 0$. Then P is absolutely normal if and only if $P_{\mathfrak{p}}$ is absolutely normal for any prime ideal $\mathfrak{p}$ which does not contain x and also for any* (*minimal*) *prime divisor $\mathfrak{p}$ of xP.*

Proof. The only if part is obvious (more generally, if a spot is not absolutely normal, then its specialization is not absolutely normal, as is easily seen). We assume that $P_{\mathfrak{p}}$ is absolutely normal for any prime $\mathfrak{p}$ as stated

above. Let I^* be a ground ring, which is a valuation ring and which dominates I'. By the assumption, for $\mathfrak{p}$ which do not contain x, we see that $(P\otimes I^*)[1/x]$ is a normal ring. On the other hand, since

$$P\otimes I^*/x(P\otimes I^*) = (P/xP)\otimes_{I/xI} I^*/xI^*,$$

$x(P\otimes I^*)$ has no imbedded prime divisor by Lemma 5.1.2. Therefore, the absolute normality of $P_{\mathfrak{p}}$ for minimal prime divisors of xP shows that for any prime divisor $\mathfrak{p}^*$ of $x(P\otimes I^*)$, $(P\otimes I^*)_{\mathfrak{p}^*}$ is a normal ring. Therefore we see that $P\otimes I^*$ is a normal ring.

LEMMA 2. *Let $\mathfrak{v}$, $\mathfrak{v}'$ and I be discrete (rank 1) valuation rings having prime elements x, x' and y respectively. Assume that I is dominated by $\mathfrak{v}$ and $\mathfrak{v}'$ and that $y\mathfrak{v}\neq x\mathfrak{v}$, $y\mathfrak{v}'\neq x'\mathfrak{v}'$. Then $\mathfrak{v}\otimes_I\mathfrak{v}'$ cannot be a normal ring.*

Proof. Let e and e' be such that $y\mathfrak{v}=x^e\mathfrak{v}$, $y\mathfrak{v}'=x'^{e'}\mathfrak{v}'$. We may assume that $e\leqq e'$. If $\mathfrak{v}\otimes\mathfrak{v}'$ is a normal ring, it is the intersection of valuation rings containing it. For any valuation v whose valuation ring contains $\mathfrak{v}\otimes\mathfrak{v}'$, $ev(x)=v(y)=e'v(x')$. Therefore $v(x)\geqq v(x')$, hence $x/x'\in\mathfrak{v}\otimes\mathfrak{v}'$, which is obviously impossible.

As a corollary to this lemma, we have

PROPOSITION 4. *If a spot P over a ground ring I is absolutely normal and if $\mathfrak{p}$ is a prime ideal of rank 1 in P such that $\mathfrak{p}\cap I\neq 0$, then* $\mathfrak{p}P_{\mathfrak{p}}=(\mathfrak{p}\cap I)P_{\mathfrak{p}}$; i.e., *$P_{\mathfrak{p}}$ is an unramified simple spot.*

THEOREM 7. *A spot P over a ground ring I is absolutely normal if (and only if) $P\otimes_I I^*$ is normal for any finite normal extension I^* of I; if I is of positive characteristic, then we may restrict I^* to purely inseparable extensions.*

Proof. By virtue of Corollary 1 to Theorem 6, we may assume that I is a valuation ring ($\neq$ a field) which is dominated by P. Let y be a prime element of I. Lemma 2 shows that y is a prime element of $P_{\mathfrak{p}}$ if $\mathfrak{p}$ is a prime ideal of rank 1 such that $\mathfrak{p}\cap I\neq 0$. Let K be any discrete valuation ring dominating I whose field of quotients is finitely generated over that of I. Let I' be a finite purely inseparable normal extension of I such that $K[I']$ is separably generated over I'. In order to prove the normality of $P\otimes K$, considering $P\otimes I'$ instead of P, we may assume that K is separably generated over I. Let p be the characteristic of I/yI. Let I^* be as follows: 1) If I/yI is perfect, then $I^*=I$. 2) When I/yI is not perfect, let $\{a_\sigma\}$ be a p-base of I/yI (i.e., a maximal set of p-independent elements of I/yI) and let b_σ be

a representative of a_σ for every σ. Then I^* is the ring generated by all p-power roots of b_σ over I (for all σ). Then I^* is a valuation ring and yI^* is the maximal ideal. Furthermore, I^*/yI^* is the smallest perfect field containing I/yI. Since I^* is algebraic over I, we see that $P\otimes I^*$ is a normal ring and by the same reason $y(P\otimes I^*)$ is semi-prime. Let K^* be an extension of K in the field generated by K and I^*. Then K^* is discrete, because of finite generation of K and by our construction of I^*. Let $\mathfrak{p}^*$ be any prime divisor of $y(P\otimes K^*)$. By Lemma 5.1.2, we see that $\mathfrak{p}^*$ is minimal. Since y is in x^*K^*, we see that

$$P\otimes K^*/x^*(P\otimes K^*) = (P\otimes I^*/y(P\otimes I^*))\otimes_{I^*/yI^*} K^*/x^*K^*.$$

This shows that $x^*(P\otimes K^*)$ is semi-prime, because I^*/yI^* is a perfect field and because $y(P\otimes I^*)$ is semi-prime. Therefore $\mathfrak{p}^*(P\otimes K^*)_{\mathfrak{p}^*}$ is generated by x^*, hence $(P\otimes K^*)_{\mathfrak{p}^*}$ is a valuation ring. Therefore we have $P\otimes K^*$ is a normal ring by Proposition 3.9, and therefore $P\otimes K$ is a normal ring. Thus the assertion is proved.

THEOREM 8. *If P and P' are absolutely normal spots over the same ground ring I and if I^* is a ground ring containing I, then $P\otimes_I P'\otimes_I I^*$ is a normal ring.*

Proof. Since every spot which as P or P' as a specialization is also absolutely normal, we may assume that I and I^* are valuation rings (or fields). 1) When I is a field, let k be the smallest perfect field containing I, then $P\otimes k$, $P'\otimes k$ are absolutely normal, and we may assume that I is perfect. Then Theorem 3.8 shows the normality of $P\otimes P'\otimes I^*$. 2) When I is not a field, let p be a prime element of I. Then by the case I a field, $(P\otimes P'\otimes I^*)[1/p]$ is a normal ring. For any prime divisor $\mathfrak{p}'$ of $p(P'\otimes I^*)$, $(P'\otimes I^*)_{\mathfrak{p}'}$ is a ground ring. Therefore for any prime divisor $\mathfrak{p}^*$ of $p(P\otimes P'\otimes I^*)$, $(P\otimes P'\otimes I^*)_{\mathfrak{p}^*}$ is a normal ring. It follows now that $P\otimes P'\otimes I^*$ is a normal ring.

COROLLARY. *If M and M' are absolutely normal models over the same ground ring, then $M\otimes M'$ is also absolutely normal.*

By the same proof as for Theorem 8, we have

PROPOSITION 5. *Let P and P' be normal spots over a ground ring I. If one of P and P' is absolutely normal and if the other is separably generated over I, then $P\otimes P'$ is a normal ring.*

5. Point set attached to an absolutely irreducible model—the notion of varieties. Let I be a ground ring and let U be a set of infinitely many algebraically independent elements over I. Then the integral closure of $I(U)$ in the algebraic closure of the field of quotients of $I(U)$ in the algebraic closure of the field of quotients of $I(U)$ is called the *universal domain* of I with respect to the set U.

We shall fix the set U and the universal domain D with respect to U, unless the contrary is explicitly stated.

A ground ring I^* which contains I is called a *canonical extension* of I if I^* is a finite integral extension of $I(U')$ with a subset U' of U. Observe that if $U - U'$ is a infinite set, then D is the universal domain of I^* with respect to $U - U'$

Now let M be an absolutely irreducible model over I. Let $\{I_\sigma ; \sigma \in \Sigma\}$ be the set of all canonical extensions of I (in D). Then the extensions $M_\sigma = M \otimes I_\sigma$ are defined and the set of M_σ forms an inverse system under projection. Let M' be the limit of this inverse system. A member of M' is called a *point* of M and M' is called the *variety* of M (with respect to U).

If P is a point of M, then there exists a uniquely determined spot P_σ in M_σ which is a representative of P in M_σ (for each $\sigma \in \Sigma$). This P_σ is called the spot of P over I_σ. When $\dim P_\sigma = n$, we say that P is a point of *dimension* n over I_σ. If $\dim P_\sigma = \dim M_\sigma$, we say that P is a *generic point* of the variety M' over I_σ.

Remark. The above notion of points does not coincide with the usual notion of points of a variety; the points in our sense contains also generic points of subvarieties over the universal domain, namely, if I is a field then M' is nothing but the extension of M over the universal domain.

Let P and Q be points of the variety M' and let P_σ and Q_σ be spots of P and Q over I_σ. Then we say that Q is a *specialization* of P over I_σ if Q_σ is a specialization of P_σ.

Let P be a spot in M. If the induced model $\phi_P(M)$ is absolutely irreducible over $\phi_P(I)$ and if $\phi_P(I) = I$, then the set of points of the variety M', whose spots over I are in $M(P)$ can be naturally identified with the variety of $\phi_P(M)$, which is called the *subvariety* of M' attached to the spot P and we say that the subvariety is defined over I. Similarly for M_σ, we defined the notion of subvarieties defined over I_σ.

Let $\mathfrak{p}$ be a prime ideal of I. If $\mathfrak{p}'$ is a prime ideal of the universal domain D such that $\mathfrak{p}' \cap I = \mathfrak{p}$, then we say that $D_{\mathfrak{p}'}$ is a universal place over $\mathfrak{p}$ (or $I_{\mathfrak{p}}$). Now, if, for a $P \in M$, $\phi_P(M)$ is absolutely irreducible over

$\phi_P(I)$ and if $\phi_P(I) \neq I$, then the following subset M'' is naturally identified with the variety of $\phi_P(M)$:

Let $I_{\mathfrak{p}}$ be the ground place dominated by P and let D' be a universal place over $I_{\mathfrak{p}}$. Let M'' be the set of points Q of M' such that i) the spot of Q over I is in $M(P)$ and ii) the spot of Q over I_σ dominates the ground place of I_σ dominated by D' (for all $\sigma \in \Sigma$).

We call M'' a *representative* of the subvariety of M' attached to P and we say that M'' is defined over $\phi_P(I)$. The union of all M'' (for all possible choice of D') is called the *subvariety* of M attached to P and we say that the subvariety is *defined* over I.

The same can be observed for M_σ.

If F is a closed set of M_σ, then the set of points whose spots over I_σ are in F is called the *closed set* of M' attached to F. For a fixed I_σ, all possible such sets are defined to be the closed sets of M' in *I_σ-topology* of M'; it will be easy to see that this really defines a topology in M'. The closed set attached to an irreducible closed set of M_σ is called a *relatively irreducible* subvariety of M' defined over I_σ.

Let L be the function field of M. Then an element f of the field of quotients of $L \otimes D$ is called a *function* on the variety M'; if f is in the field of quotients of $L \otimes I_\sigma$, then we say that f is *defined* over I_σ. A function f which is defined over I_σ is said to be regular at a point P if f is in the spot of P over I_σ (observe that this is independent of I_σ whenever f is defined). P defined a uniquely determined homomorphism $\lim \phi_{P_\sigma}$, which is denoted by ϕ_P. Then $\phi_P(f)$ is defined if and only if f is regular at P; in this case $\phi_P(f)$ is called the *value* of f at P.

When M is an affine model defined by the affine ring $I[x_1, \cdots, x_n]$, the system $(x_1, \cdots, x_n)$ is called a *coordinate system* of the variety M'. If P is a point of M', then the x_i is regular at P. The system $(\phi_P(x_1), \cdots, \phi_P(x_n))$ is called the coordinates of P attached to the coordinate system $(x_1, \cdots, x_n)$.

When M is a projective model defined by homogeneous coordinates $(z_0, \cdots, z_n)$, we say that $(z_0, \cdots, z_n)$ is a *homogeneous coordinate system* of M' and we say that M' is a *projective variety*. If P is a point of M', then there exists one i such that z_j/z_i is regular at P for every j. Then, t being an arbitrary non-zero element of the set of values of regular functions at P, the system $(t\phi_P(z_0/z_i), \cdots, t\phi_P(z_n/z_i))$ is called *homogeneous coordinates* of P.

Let P be a point of the variety M' and let P_σ be the spots of P over I_σ. (1) P is called a *simple point* of M' if every P_σ is a simple spot; otherwise,

P is called a *singular* point. (2) P is called a *normal point* if every P_σ is a normal spot.

THEOREM 9. *With the same notations as above, the following three conditions are equivalent to each other:*

(1) *P is a normal point.*

(2) *Every P_σ is absolutely normal.*

(3) *There exists one I_σ such that P_σ is absolutely normal.*

Proof. Let I be I_0. i) Assuming (1), we shall prove (2). It will be sufficient to show that P_0 is absolutely normal. If P_0 is not absolutely normal, then there exists a finite normal extension I_σ of I such that one extension of P_0 over I_σ is not normal. Since I_σ is a normal extension of I, extensions of P_0 over I_σ are conjugate to each other, hence P_σ is not normal, which is a contradiction. Thus P_0 is absolutely normal, and every P_σ is absolutely normal. ii) It is obvious that (3) follows from (2). iii) Assuming (3), we shall prove (1). It is obvious that if $I_{\sigma'}$ contains I_σ, then $P_{\sigma'}$ is also absolutely normal. Since for any given $I_{\sigma''}$, there exists an $I_{\sigma'}$ which contains both I_σ and $I_{\sigma''}$, we see that $P_{\sigma''}$ is a normal spot, which shows that P is a normal point. Thus the theorem is proved.

Similar assertion for simple points will be proved in the next chapter.

6. The order of inseparability. Throughout this section, let L be a function field over a ground field k, let $(x) = (x_1, \cdots, x_n)$ be an arbitrary transcendence base of L over k, let $\mathfrak{o}$ be a local ring of rank zero which contains k, let $\mathfrak{n}$ be the maximal ideal of $\mathfrak{o}$, let k' be the residue class field $\mathfrak{o}/\mathfrak{n}$ and let $(u) = (u_1, \cdots)$ be a transcendence base of k' over k.

Obviously, $L \times_k \mathfrak{o}$ is the direct sum of a finite number of local rings of rank zero. When a local ring $\mathfrak{S}$ is a direct summand of $L \times_k \mathfrak{o}$, then the length $l(\mathfrak{S})$ of $\mathfrak{S}$ is denoted by $i(L/k; \phi_{\mathfrak{S}}(\mathfrak{S})/\mathfrak{o})$. If $i(L/k; \phi_{\mathfrak{S}}(\mathfrak{S})/\mathfrak{o})$ does not depend on the choice of direct summand, it is called the *order of inseparability* of L over $\mathfrak{o}$ and is denoted by $i(L/k; \mathfrak{o})$. Observe that if $\mathfrak{o}$ is the algebraic closure of k, then $i(L/k; \mathfrak{o})$ is the *order of inseparability* $[L : k]_i$ of L (in the sense of Weil [6]), as is easily seen by the following

THEOREM 10. *$\mathfrak{S}' = \mathfrak{S}/\mathfrak{n}\mathfrak{S}$ is a direct summand of $L \times_k k'$, and*

$$i(L/k; \phi_{\mathfrak{S}}(\mathfrak{S})/\mathfrak{o}) = l(\mathfrak{o}) \cdot i(L/k;\ \phi_{\mathfrak{S}}(\mathfrak{S})/k').$$

Furthermore,

$$i(L/k; \phi_{\mathfrak{S}}(\mathfrak{S})/k') = [L : k(x)]_i / [\phi_{\mathfrak{S}}(\mathfrak{S}) : k'(x)]_i.$$

On the other hand, if P is a spot over a ground place I, $\mathfrak{m}$ and $\mathfrak{n}'$ are maximal ideals of P and I respectively, I^* is a ground ring containing I, P' is a component of $P \times_I I^*$, $P/\mathfrak{m} = L$, $I/\mathfrak{n}' = k$, $\mathfrak{o} = I^*/\mathfrak{n}'I^*$ and $\mathfrak{S} = P'/\mathfrak{n}'P'$, then

$$i(L/k; \phi_{\mathfrak{S}}(\mathfrak{S})/\mathfrak{o}) = e(\mathfrak{m}P')/e(\mathfrak{m}).$$

Proof. The first formula can be proved easily by induction on the length of $\mathfrak{o}$. We shall prove now the second formula. Since $L \times_k k' = L(u) \times_{k(x,u)} K(x)$ and since $[L : k(x)]_i = [L(u) : k(x,u)]_i$, we may assume that L and k' are algebraic over k. Let k'' be the maximal separable extension of k contained in k' and let L'' be the direct summand of $L \times k''$ contained in $\mathfrak{S}'$. Then L'' is a field and is separable over L. Therefore, considering L'' and k'' instead of L and k, we may assume that k' is purely inseparable over k. By the same reason, we may assume furthermore that L is purely inseparable over k, and we have a proof of this case easily. Lastly we shall prove the third formula. By the same reason as above (observing residue class fields), we may assume that the residue class fields of P, I^* are algebraic over that of I. On the other hand, we may assume that I^* is also a ground place dominated by P'. Then extending P and P' over the completions of I and I^* respectively, we may assume that I and I^* are complete (observe that the invariance of the multiplicities under the extensions follows from Proposition 2 in Appendix 2 in Part II of the present sequence of papers). Again by the same reason as for the second formula, we may assume that the residue class fields of P and I^* are purely inseparable over that of I. Now, if the residue class field k' of I^* is finite over $k = I/\mathfrak{n}'$, namely, if I^* is finite over I (because I is complete), then by the extension formula for multiplicity, we see that $e(\mathfrak{m}P')/e(\mathfrak{m}) = [L : k]/[\phi_{P'}(P') : k']$, hence by the second formula we have the required formula. Now the general case (where I^* is not finite over I) can be reduced easily from the case where I^* is finite over I.

COROLLARY. *Under the same notations as above,* (i) *if $\mathfrak{o}$ is a field and if one of L and $\mathfrak{o}$ is separably generated over k then* $i(L/k; \mathfrak{o}) = 1$ *and* (ii) *L is separably generated over k if and only if* $[L : k]_i = 1$.

By virtue of the first part of Theorem 10, we see that the case where $\mathfrak{o}$ is a field is fundamental and we shall consider this case.

THEOREM 11. *Let L' be a function field over k containing L and let k'' be a field containing k'. Assume that they are contained in a field and that* $\dim_k L' = \dim_{k''} k''(L')$. *Then we have*

(i) $i(L/k\,;k''(L)/k'')=i(L/k\,;k'(L)/k')\cdot i(k'(L)/k'\,;k''(L)/k'')$,

(ii) $i(L'/k\,;k'(L')/k')=i(L/k\,;k'(L)/k')\cdot i(L'/L\,;k'(L')/k'(L))$.

Proof. We shall prove (i) first. Let (u'') be a transcendence base of k'' over k'. Then by the same reason as in the proof of Theorem 10, we may consider $k(u,u'')$ and $k'(u'')$ instead of k and k' respectively and we may assume that k'' is an algebraic extension of k. Similarly, we may assume that L is algebraic over k. Then just as in the proof of Theorem 10, we may assume that L, k' and k'' are purely inseparable over k. Then, in this case, $i(L/k\,;k''(L)/k'')=l(L\otimes_k k'')$, $i(L/k\,;k'(L)/k')=l(L\otimes_k k')$ and $i(k'(L)/k'\,;k''(L)/k'')=l(k'(L)\otimes_{k'}k'')$. Therefore the formula follows from the fact that $L\otimes_k k''=(L\otimes_k k')\otimes_{k'}k''$. Now we consider (ii). Just as above, we may assume that k' and L' are purely inseparable over k. Then we prove the formula by the fact that $L'\otimes_k k'=L'\otimes_L(L\otimes_k k')$.

COROLLARY. *If furthermore L'' is a field containing $k'(L)$ and if $\dim_L L'=\dim_{L''}L'(L'')$ and $k'(L')\subseteq L'(L'')$, then*

$$i(L'/k\,;k'(L')/k')/i(L/k\,;k'(L)/k')$$
$$=i(L'/L\,;L'(L'')/L'')/i(k'(L')/k'(L)\,;L'(L'')/L'').$$

In particular,

$$[L':k]_i/[L:k]_i=[L':L]_i/[\bar{k}(L'):\bar{k}(L)]_i,$$

where $\bar{k}$ is the algebraic closure of k.

Proof. Both sides of the first formula are equal to $i(L'/L\,;k'(L')/k'(L))$. The second formula follows from the first one as the special where $k'=\bar{k}$ and L'' is the algebraic closure of L.

As applications of our treatment, we shall show how Propositions 29-31 in Weil [6, Chapter I] can be proved.

PROPOSITION 6. *Let L be a function field over a field k of characteristic $p\neq 0$. If m is the natural number such that $[L:k]_i=p^m$ $(\neq 1)$, then $i(L/k\,;k^{1/p})>1$ (hence $\geqq p$) and $L(k^{p^{-m}})$ is separably generated over $k^{p^{-m}}$, i.e., $i(L/k\,;k^{p^{-m}})=p^m$* (Weil [6]).

Proof. Since L is not separably generated over k, $L\otimes k^{1/p}$ is not an integral domain and therefore $i(L/k\,;k^{1/p})>1$. Hence

$$[L(k^{1/p}):k^{1/p}]_i=[L:k]_i/i(L/k\,;k^{1/p})=p^n \text{ with } n<m.$$

Therefore we have $[L(k^{p^{-m}}):k^{p^{-m}}]_i=1$, $i(L/k\,;k^{p^{-m}})=p^m$.

PROPOSITION 7. *Assume that* $L=k(x_1,\cdots,x_r,x_{r+1})$ *and that* $\dim_k L=r$. *Assume furthermore that* $[L:k]_i=p^m$. *Let* $\bar{k}$ *be the algebraic closure of* k *and let* F, G, H *be the irreducible polynomials in indeterminates* $X_1,\cdots,X_{r+1}$ *for* $(x_1,\cdots,x_{r+1})$ *over* k, $k^{p^{-m}}$, $\bar{k}$ *respectively. Then we have, except for constant factors,* $F=G^{p^m}$ *and* $G=HH'$ *with a polynomial* H' *over* $\bar{k}$ *such that* $H'(x_1,\cdots,x_{r+1})\neq 0$ (Weil [6]).

Proof. Set $A=k[X_1,\cdots,X_{r+1}]$. Then A/FA is an affine ring of L. By Proposition 6, $l(L\otimes k^{p^{-m}})=p^m$. This shows that, denoting by A' the ring $A[k^{p^{-m}}]$, we have $FA'=G^{p^m}A'$ because $k^{p^{-m}}$ is purely inseparable over k. As for the last assertion, we have, denoting by $\bar{A}$ the ring $A[\bar{k}]$, $G\bar{A}$ is semi-prime because $[L(k^{p^{-m}}):k^{p^{-m}}]_i=1$.

In order to apply our observation in this section to absolutely irreducible models, we shall introduce another notation as follows:

Let P be a spot over a ground ring I and assume that P is a regular extension of I. Let I' be a ground ring containing I. Assume that P' is a component of $P\times_I I'$. Let $\mathfrak{m}$ be the maximal ideal of P. Then $i(P/I;P'/I')$ denotes the length $l(P'/\mathfrak{m}P')$, which is equal to $e(\mathfrak{m}P')/e(\mathfrak{m})$ (by Theorem 10) and also to $i((P/\mathfrak{m})/(I/(\mathfrak{m}\cap I))\,;\,(\phi_{P'}(P'))/(I'_{\mathfrak{p}'}/(\mathfrak{m}\cap I)I'_{\mathfrak{p}'})$ where $I'_{\mathfrak{p}'}$ is the ground place of I' dominated by P'.

7. The definition of cycles. Let M be an absolutely irreducible model over a ground ring I. An element Z of the free module generated by all spots of M over the field of rational numbers is called a *generalized cycle* on M. For a generalized cycle $Z=\sum c_iP_i$ $(P_i\in M)$, (i) each P_i whose coefficient c_i is different from zero is called a *component* of Z and (ii) the union of the loci of the components of Z is called the *carrier* of Z and is denoted by $\operatorname{Supp} Z$.

A generalized cycle Z is said to be *effective*, and is denoted by $Z\succ 0$, if the coefficient of every component of Z is positive (Z may be zero). For two generalized cycles Z and Z', if $Z-Z'$ is effective, then we write $Z\succ Z'$. Then the relation $\succ$ gives a partial order in the group of generalized cycles on a model.

For a generalized cycle Z, there are effective generalized cycles Z_+ and Z_- such that 1) $Z=Z_+-Z_-$ and 2) Z_+ and Z_- have no common components. Such Z_+ and Z_- are uniquely determined. Z_+ and Z_- are called the *positive part* and the *negative part* of Z respectively.

An *integral cycle* is a generalized cycle whose coefficients are all integers. A *cycle* is an integral cycle whose components are absolutely simple. A cycle

(or a generalized cycle or an integral cycle) is called an *r-cycle* (or a generalized r-cycle or an integral r-cycle) if its components are all of dimension r; it is said to be *unmixed* if furthermore all components dominate the same ground place.

A $(\dim M - 1)$-cycle on M is called a *divisor*; a *generalized divisor* and an *integral divisor* are defined similarly.

A spot in M can be regarded as a generalized cycle (an integral cycle) on M; it is said to be *prime*; thus we define generalized prime cycles, prime cycles, prime divisors and so on.

Let I^* be a ground ring containing I. For a generalized cycle $Z = \sum c_j P_j$ on M, let $\sigma(Z)$ be $\sum c_j \cdot i(P_j/I; P^*_{jl}/I^*) P^*_{jl}$, where P^*_{jl} runs over all components of $P_j \times I^*$ for each j. This σ defines a homomorphism from the group of generalized cycles on M into that of $M \otimes I^*$, which will be denoted by $\sigma_{I^*/I}$. Observe that if I^* is a canonical extension of I then $\sigma_{I^*/I}$ is an isomorphism. By our definition, it follows easily from Theorems 10 and 11 that

THEOREM 12. *If I, I^* and I^{**} are ground rings such that $I \subseteq I^* \subseteq I^{**}$, then $\sigma_{I^{**}/I} = \sigma_{I^{**}/I^*} \cdot \sigma_{I^*/I}$.*

Observe that $\sigma_{I^*/I}$ preserves the properties of being an integral cycle, of being a cycle, and so on. Therefore, when we consider canonical extensions of I, cycles, generalized cycles, integral cycles, and so on, on M can be identified with those on the extensions of M, and therefore we can define generalized cycles, cycles, r-cycles, integral cycles on the variety of M with respect to a universal domain, hence we can define also I-rational cycles, I-rational integral cycles, and so on, on the variety.

If M' is a model contained in M, then the following mapping ϕ is a homomorphism from the group of generalized cycles on M onto that of M': $\phi(\sum c_i P_i) = \sum_{P_i \in M'} c_i P_i$. This mapping ϕ is called the *restriction* on M' and is denoted by $[\ ]_{M'}$, namely,

$$[\sum c_i P_i]_{M'} = \sum_{P_i \in M'} c_i P_i.$$

8. Divisor of a function. We shall define at first the notion of algebraic projection in a special case (the general definition will be given in Part IV of the present sequence of papers).

Let M be a model of a function field L over a ground ring I and let L' be a finite algebraic extension of L. Assume that a model M' of L' dominates M and is dominated by the derived normal model $N(M; L')$ of M in L' and that L' is a regular extension of I (hence L is also a regular extension of I).

The *algebraic projection* $\mathrm{pr}^{M'}{}_M$ (or pr) from M' into M is defined by the following conditions: (1) If $P' \in M'$, then, denoting by P the projection of P' on M, $\mathrm{pr}\, P' = [\phi_{P'}(P') : \phi_P(P)]P$ (this is an integral cycle on M) and (2) pr is a homomorphism from the group of generalized cycles on M' into that on M. From this definition, it follows immediately that

PROPOSITION 8. *If* $\mathrm{pr}^{M''}{}_{M'}$ *and* $\mathrm{pr}^{M'}{}_M$ *are defined, then* $\mathrm{pr}^{M''}{}_M$ *is also defined and coincides with* $\mathrm{pr}^{M''}{}_{M'} \cdot \mathrm{pr}^{M'}{}_M$.

Now let M, L and I be as above. Let $f \neq 0$ be a function on M (i.e., $f \in L$) and let $N(M)$ be the derived normal model of M. For any spot P' of rank 1 in $N(M)$, let $v_{P'}$ be the normalized valuation defined by P'. Then $(f)_{N(M)} = \sum v_{P'}(f)P'$ (P' runs over all spots P' of rank 1 in $N(M)$) is an integral divisor on $N(M)$. The algebraic projection of this integral divisor $(f)_{N(M)}$ on M is called the *generalized divisor* of the function f on M and is denoted by $(f)_M$ or merely by (f). If Z_0 and Z_∞ are the positive and negative parts of $(f)_{N(M)}$ respectively, then $\mathrm{pr}\, Z_0$ and $\mathrm{pr}\, Z_\infty$ are called the generalized *zero-divisor* and the generalized *pole-divisor* of f on M and they are denoted by $(f)_{0M}$ and $(f)_{\infty M}$ or merely by $(f)_0$ and $(f)_\infty$ respectively. Since $(f)_{N(M)} = Z_0 - Z_\infty$, we have $(f)_M = (f)_{0M} - (f)_{\infty M}$. Though Z_0 and Z_∞ have no common component, $(f)_{0M}$ and $(f)_{\infty M}$ may have some common components.

THEOREM 13. *Let M, f and I be as above and let I^* be a ground ring containing I. Then*

$$\begin{aligned} &\sigma_{I^*/I}((f)_M) = (f)_{M\otimes I^*}, \\ (1) \qquad &\sigma_{I^*/I}((f)_{0M}) = (f)_{0,M\otimes I^*}, \\ &\sigma_{I^*/I}((f)_{\infty M}) = (f)_{\infty,M\otimes I^*}. \end{aligned}$$

If, for any spot P of rank 1 *in M, either f or f^{-1} is in P, then*

$$(2) \qquad \begin{aligned} &(f)_{0M} = \sum e(fP)P \quad \text{(where } P \text{ runs over all spots of rank 1 in } M \text{ which contains } f), \\ &(f)_{\infty M} = \sum e(f^{-1}P)P \quad \text{(where } P \text{ runs over all spots of rank 1 in } M \text{ which contains } f^{-1}). \end{aligned}$$

Proof. (2) is an immediate consequence of the extension formula of multiplicity (see [3]). Applying (2) to $N(M) \otimes I^*$, we have

$$(f)_{0,N(M)\otimes I^*} = \sum e(fP^*)P^* \quad (f \in P^*, \operatorname{rank} P^* = 1, P^* \in N(M) \otimes I^*)$$

and

$$(f)_{\infty,N(M)\otimes I^*} = \sum e(f^{-1}P^*)P^* \quad (f^{-1} \in P^*, \operatorname{rank} P^* = 1, P^* \in N(M) \otimes I^*).$$

For an arbitrary spot P^* of rank 1 in $N(M)\otimes I^*$ which contains f, let P be the spot of rank 1 in $N(M)$ which is dominated by P^*. Then the coefficient c of P^* in $\sigma_{I^*/I}((f)_{N(M)})$ is equal to $v_P(f)\cdot i(P/I;P^*/I^*)$ by the definition of $\sigma_{I^*/I}$. Since P is a valuation ring, we have easily $c=e(fP^*)$ and we have $\sigma_{I^*/I}((f)_{0,N(M)})=(f)_{0,N(M)\otimes I^*}$. Similarly we have

$$\sigma_{I^*/I}((f)_{\infty,N(M)})=(f)_{\infty,N(M)\otimes I^*}.$$

Therefore (1) follows from the following

PROPOSITION 9. $\mathrm{pr}^{M'\otimes I^*}{}_{M\otimes I^*}\cdot\sigma_{I^*/I}=\sigma_{I^*/I}\cdot\mathrm{pr}^{M'}{}_M$, *where M, M', I and I^* are as above.*

Proof. It is sufficient to prove $\mathrm{pr}\cdot\sigma_{I^*/I}P'=\sigma_{I^*/I}\cdot\mathrm{pr}\,P'$ for an arbitrary spot $P'\in M'$. $\sigma_{I^*/I}P'=\sum i(P'/I;P^*_j/I^*)P^*_j$, where P^*_j runs over all components of $P'\times I^*$. Let $\mathfrak{m}'$ be the maximal ideal of P'. Then $i(P'/I;P^*_j/I^*)=l(P^*_j/\mathfrak{m}'P^*_j)$, hence $\mathrm{pr}\cdot\sigma_{I^*/I}P'=\sum_j l(P^*_j/\mathfrak{m}'P^*_j;\mathrm{proj}\,P^*_j)\,(\mathrm{proj}\,P^*_j)$. This shows that, if $P_1,\cdots,P_r$ are components of $\mathrm{pr}\cdot\sigma_{I^*/I}P'$, then $\mathrm{pr}\cdot\sigma_{I^*/I}P'=\sum_j l((P'\times I^*)_{S(j)};P_j)P_j$, where $S(j)$ is the intersection of complements of maximal ideals in $P'\times I^*$ which lie over that of P_j. On the other hand, if we denote by P and $\mathfrak{m}$, $\mathrm{pr}\,P'$ and its maximal ideal, then

$$\begin{aligned}\sigma_{I^*/I}\cdot\mathrm{pr}\,P'&=[\phi_{P'}(P'):\phi_P(P)]\cdot(\textstyle\sum i(P/I;P_j/I^*)P_j)\\&=\textstyle\sum l(P_j/\mathfrak{m}P_j)\cdot[\phi_{P'}(P'):\phi_P(P)]\cdot P_j.\end{aligned}$$

Now we see the equality of these two generalized cycles easily.

9. Ideals on a model. An *ideal* $\mathfrak{A}$ of a model M is a set of ideals $\mathfrak{a}(P)$ of spots $P\in M$ ($\mathfrak{a}(P)$ may be equal to P) such that for any spot $P\in M$, there exists an affine model A contained in M and containing P which satisfies the following condition: There is an ideal $\mathfrak{a}$ of the affine ring $\mathfrak{o}$ of A such that $\mathfrak{a}(Q)=\mathfrak{a}Q$ for any spot $Q\in A$. $\mathfrak{a}(P)$ is called the *P-component* of $\mathfrak{A}$ and the ideal $\mathfrak{a}$ above is called the *affine representative* of $\mathfrak{A}$ in A (or in $\mathfrak{o}$).

This definition shows that for an ideal $\mathfrak{A}$ of a model M, there exist a finite number of affine models $A_1,\cdots,A_n$ such that 1) M is the union of the A_i and 2) $\mathfrak{A}$ has the affine representative in each of the A_i.

Sums, products and intersections of ideals of a model are defined by component-wise operations.

If $\mathfrak{A}$ is an ideal of a model M, the set of spots $P\in M$, such that the P-component of $\mathfrak{A}$ is different from P, forms obviously a closed set of M, which is called the *closed set* defined by the ideal $\mathfrak{A}$.

We say that an ideal in a model M is *prime* or *primary* if 1) the closed set defined by the ideal is irreducible and 2) every component is prime or primary respectively. Then we see the decomposition theorems to intersections of primary ideals as in the case of Noetherian rings.

Remark. If $\mathfrak{a}$ is an ideal of an affine ring $\mathfrak{o}$, then $\mathfrak{a}$ defines an ideal $\mathfrak{A}$ of the affine model A of $\mathfrak{o}$ such that the P-component of $\mathfrak{A}$ is $\mathfrak{a}P$ for any $P \in A$. If $\mathfrak{b}$ is a homogeneous ideal of a homogeneous coordinate ring of a projective model M, then $\mathfrak{b}$ defines an ideal $\mathfrak{B}$ of M as follows: Let $\mathfrak{h} = I[z_0, \cdots, z_n]$ be the homogeneous coordinate ring (z_i are homogeneous elements of degree 1). If a spot P is in the affine model of $I[z_0/z, \cdots, z_n/z]$ (z being a homogeneous element of degree 1), then let $\mathfrak{b}(z)$ be the set of elements of the form b/z^r with homogeneous element b of degree r (r being arbitrary). Then $\mathfrak{b}(z)$ is an ideal of $I[z_0/z, \cdots, z_n/z]$ and $\mathfrak{b}(z)P$ is uniquely determined (independently on the choice of z). Now, $\mathfrak{B}$ is defined so that the P-component is $\mathfrak{b}(z)P$.

Ideals $\mathfrak{A}$ and $\mathfrak{B}$ obtained as above are called *ordinary ideals.*

Let $\mathfrak{A}$ be an ideal of an absolutely irreducible model M and let F be the closed set defined by $\mathfrak{A}$. Let $P_1, \cdots, P_n$ be the generating spots of the irreducible components of F and let $\mathfrak{a}_i$ be the P_i-component of $\mathfrak{A}$ for each i. Then $\mathfrak{a}_i$ is a primary ideal belonging to the maximal ideal of P_i. Then $\mathfrak{A}$ defines two integral cycles $\sum l(P_i/\mathfrak{a}_iP_i)P_i$ and $\sum e(\mathfrak{a}_i)P_i$; they are denoted by $Z_1(\mathfrak{A})$ and $Z_2(\mathfrak{A})$ respectively. $\max(\operatorname{rank} P_1, \cdots, \operatorname{rank} P_n)$ and $\min(\operatorname{rank} P_1, \cdots, \operatorname{rank} P_n)$ are called *maximal rank* and *minimal rank of* $\mathfrak{A}$ respectively, and denoted by max-rank $\mathfrak{A}$ and min-rank $\mathfrak{A}$. If max-rank $\mathfrak{A}$ = min-rank $\mathfrak{A}$, then this number is called the *rank* of $\mathfrak{A}$ and is denoted by rank $\mathfrak{A}$.

If rank $\mathfrak{A}$ is defined and if every component of $\mathfrak{A}$ is generated by rank $\mathfrak{A}$ elements, then $\mathfrak{A}$ is said to be a *principal ideal* of rank (rank $\mathfrak{A}$). Observe that if $\mathfrak{A}$ is a principal ideal of rank r and if each P_i defined above has a distinct system of parameters, then $Z_1(\mathfrak{A}) = Z_2(\mathfrak{A})$. In particular, if $\mathfrak{A}$ is a principal ideal of rank 1, then $Z_1(\mathfrak{A}) = Z_2(\mathfrak{A})$, which is called the *effective principal divisor* defined by $\mathfrak{A}$. In general, if $Z_1(\mathfrak{A}) = Z_2(\mathfrak{A})$ for an ideal $\mathfrak{A}$, the integral cycle $Z_1(\mathfrak{A})$ is called the *integral cycle defined by* $\mathfrak{A}$ and is denoted by $Z(\mathfrak{A})$.

If D is an effective principal divisor (defined some principal ideal of rank 1) on a model M, then for every spot P of M there exists an affine model A contained in M and containing P such that $[D]_A = (f_A)_A$ for an element f_A of the affine ring of A. Such an f_A is called a *local equation* of D at P, or on A.

PROPOSITION 10. *If $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are principal ideals of rank* 1 *on M, then $\mathfrak{A}_1\mathfrak{A}_2$ is also principal and $Z(\mathfrak{A}_1\mathfrak{A}_2) = Z(\mathfrak{A}_1) + Z(\mathfrak{A}_2)$. Furthermore, if D is an integral divisor contained in the group generated by all effective principal divisors, then for any spot $P \in M$, there exists an affine model A contained in M and containing P such that $[D]_A = (f_A)_A$ with a function f_A on M.*

Proof. Easy.

f_A as above is called the *local equation* of D at P, or on A. The group generated by all effective principal divisors is called the *principal divisor group*, and elements of the group are called *principal divisors.*

On the other hand, let I^* be a ground ring containing I. Then for an ideal $\mathfrak{A}$ of M, the ideal $\mathfrak{A}^*$ of $M \otimes I^*$ defined as follows is called the *extension* of $\mathfrak{A}$ over I^*, or in $M \otimes I^*$, and is denoted by $\mathfrak{A} \otimes I^*$: For an arbitrary spot $P^* \in M \otimes I^*$, let P be the spot in M which is dominated by P^* (i.e., P^* is an extension of P), and let $\mathfrak{a}(P)$ be the P-component of $\mathfrak{A}$. Then the P^*-component of $\mathfrak{A}^*$ is defined to be $\mathfrak{a}(P)P^*$. More generally, if M' is a model dominating M, then we define the *extension* of $\mathfrak{A}$ in M' by the same way. By our definition, we see easily the following

THEOREM 14. $\sigma_{I^*/I}(Z_i(\mathfrak{A})) = Z_i(\mathfrak{A} \otimes I^*)$ *for* $i = 1, 2$,

Proof. As for $i = 1$, the proof is easy. Since $\sigma_{I^*/I}(Z_1(\mathfrak{A}^n)) = Z_1(\mathfrak{A}^n \otimes I^*)$ for any $n = 1, 2, 3, \cdots$, we see the equality in the case where $i = 2$.

10. Linear equivalence classes of divisors and line bundles. We say that two generalized divisors D and D' on a model M are *linearly equivalent* to each other if there exists a function f on M such that $D - D' = (f)$; in this case, we write $D \sim D'$.

PROPOSITION 11. *If I is a Dedekind domain and if u is a transcendental element over I, then $I(u)$ is a unique factorization ring.*

Proof. Let $\mathfrak{p}^*$ be any prime ideal of rank 1 in $I(u)$ and set $\mathfrak{p} = I \cap \mathfrak{p}^*$. Then we have $\mathfrak{p}^* = \mathfrak{p}I(u)$. Since I is a Dedekind domain, $\mathfrak{p}$ is generated by two elements, say, a and b. Set $x = au + b$. Then we see that $\mathfrak{p}^*$ is the unique prime ideal of $I(u)$ containing x and that $xI(u)_{\mathfrak{p}^*} = \mathfrak{p}I(u)_{\mathfrak{p}^*} = \mathfrak{p}^*I(u)_{\mathfrak{p}^*}$, which shows that $xI(u) = \mathfrak{p}^*I(u)$. Therefore $I(u)$ is a unique factorization ring.

THEOREM 15. *Let D be a principal divisor on a model M of a function field L over a ground ring I and let I^* be a canonical extension of I (with respect to a universal domain).*

(1) *If there exists a function f on $M \otimes I^*$ such that $(f)_{M\otimes I^*} \succ \sigma_{I^*/I}(D)$, then f can be expressed in the form $\sum_1^n c_i w_i$ with $c_i \in I^*$ and $w_i \in L$ such that $(w_i)_M \succ D$ for each i.*

(2) *if there exists a function f on $M \otimes I^*$ such that $(f)_{M\otimes I^*} = \sigma_{I^*/I}(D)$ and if M is complete, then there exists a function g on M such that* (i) *$g/f \in I^*$ and* (ii) *$(g)_M - D$ is an effective principal divisor with local equations in I.*

(3) *Besides the conditions in* (2), *if I is a unique factorization ring, then we can choose g so that $(g)_M = D$, namely, D is linearly equivalent to zero on M.*

Proof. We shall prove (1) first. We may assume that I^* is finitely generated over I, namely, there are a finite number of algebraically independent elements $u_1, \cdots, u_n$ over I such that I^* is a finite integral extension of $I(u_1, \cdots, u_n)$. We shall use induction on n. Assume that $n \geqq 1$. Then set $I' = I(u_1)$, $D' = {}_{I'/I}(D)$. Then by induction on $n \cdot f = \sum c_i w'_i$ with $c_i \in I^*$ and $w'_i \in L(u_1)$ such that $(w'_i)_{M\otimes I'} \succ D'$. Therefore if we know that the assertion is true for the cases where $n = 0$ and $I^* = I(u)$ with a transcendental element u, then we see that the assertion is true in every case. Thus we shall consider only these cases. We shall show at first that $f \in L \otimes I^*$. This is obvious if I^* is integral over I and therefore we assume that $I^* = I(u)$. Since $L[u]$ is a unique factorization ring, we see that $L \otimes I^*$ is a unique factorization ring. Therefore we see easily that if f is not in $L \otimes I^*$, then there exists a simple spot P^* of rank 1 in $M \otimes I^*$ such that (i) $L \otimes I^* \subseteq P^*$ and (ii) f^{-1} is a non-unit in P^*. Then the negative component of (f) has P^* as a component and we have a contradiction to the assumption that $(f) \succ \sigma_{I^*/I}(D)$. Thus $f \in L \otimes I^*$. Let $A_1, \cdots, A_m$ be affine models contained in M such that M is the union of the A_i and that D has a local equation f_i on A_i for each i. If P is an arbitrary spot of rank 1 in A_i, then f/f_i is in every component of $P \times I^*$ and therefore $f/f_i \in P \otimes I^*$ by Theorem 2. Therefore there are a finite number of elements $c_1, \cdots, c_r$ of I^* such that $f/f_i \in P \otimes (\sum Ic_j)$ for any i and P. Since I is a Dedekind domain, there is a module $\mathfrak{M}$ such that $\mathfrak{M} + \sum Ic_j$ (direct sum) is a free module over I. Let $e_1 + m_1, \cdots, e_s + m_s$ ($e_i \in \sum Ic_j$, $m_i \in \mathfrak{M}$) be a linearly independent base of the direct sum. Then f is expressed uniquely in the form $\sum_1^s w_i(e_i + m_i)$ with $w_i \in L$ and that $f/f_i \in P \otimes ((\sum Ic_j) + \mathfrak{M})$ shows that $w_i/f_i \in P$. There-

fore for every w_i which is not zero, we have $(w_i)_M \succ D$. Thus (1) is proved. Now we shall prove (2). By virtue of (1), there exists an element g of L such that $(g)_M \succ D$. It follows that $(g/f)_{M\otimes I^*} \succ 0$. Since M is complete, $M \otimes I^*$ is also complete, hence $g/f \in I^*$. Since $(g/f)_{M\otimes I^*} = \sigma_{I^*/I}((g)_M - D)$, the remaining part of (2) follows easily from (3), considering local models attached to ground places. Therefore we shall prove (3) lastly. Since $I(u_1, \cdots, u_n)$ is also a unique factorization ring (for algebraically independent elements u_i), we can reduce to the case where I^* is either a finite integral extension of I or of the form $I(u)$, using induction as in the proof of (1). On the other hand, by the proof of (2) above, we may assume that $f \in I^*$. Let g be a generator of $fI^* \cap I$. It is sufficient to prove that $fI^* = gI^*$. This is easy if $I^* = I(u)$ and therefore we assume that I^* is an integral extension of I. Now, in the proof of (1), since I^* is a free module over I, if we take a linearly independent base $c_1, \cdots, c_r$ of I^* over I, we see that $f = \sum c_i w_i$ with $w_i \in L$ such that $(w_i) \succ D$ for any $w_i \neq 0$. Since the w_i are unique and since $f \in I^*$, we have $w_i \in I$ and therefore $fI^* = \sum w_i I^*$ and therefore g is a generator of $\sum w_i I$ and $fI^* = gI^*$. Thus the proof of Theorem 15 is completed.

We say that a fibre bundle W with base M and fibre F is a *line bundle* if (1) F is the affine model of $I[x]$ (I is the ground ring of M, x is a transcendental element over I), which is called a model of the affine 1-space over I (see Chapter 6, § 1) and (2) for the x above, every transition mapping σ has the property that $\sigma(x)/x$ is in the function field of M.

Now let G_1 be the principal divisor group of a model M of a function field L over a ground ring I. The set of general divisors which are linearly equivalent to zero form a subgroup of G_1, which will be denoted by G_2 in this section.

Let D be an element of G_1. Then there are affine models $A_1, \cdots, A_n$ such that 1) the union of the A_i is M and 2) D has a local equation f_i on each A_i. Then f_i/f_j is a unit in $A_i \cap A_j$. Let x be a variable over L and let σ_i be the automorphism of $L(x)$ such that $\sigma_i(x) = f_i x$. Then defining σ_i to be the transition mapping on A_i, we have a line bundle, which is called a *line bundle defined by* D and is denoted by $W(D)$. It is easy to see that $W(D)$ is unique to within equivalence of line bundles.

THEOREM 16. *Let D and D' be elements of G_1 above. Then $W(D)$ and $W(D')$ are equivalent to each other if and only if D and D' are linearly equivalent to each other. Furthermore, any line bundle is equivalent to some $W(D)$ $(D \in G_1)$ and therefore there is a one to one correspondence between G_1/G_2 and the set of equivalence classes of line bundles.*

Proof. If D and D' are linearly equivalent to each other, then there exists an element f of L such that $D - D' = (f)$. Therefore when g_i's are local equations for D', fg_i's are local equations for D and we see the equivalence between $W(D)$ and $W(D')$. Conversely, assume that $W(D)$ and $W(D')$ are equivalent to each other. Let $A_1, \cdots, A_n$ be affine models such that $M = \cup_i A_i$ and D and D' have local equations g_i and g'_i on each A_i. We may assume that $W(D)$ and $W(D')$ are defined by these local equations. Since $W(D)$ and $W(D')$ are equivalent to each other, there exists an automorphism σ of $L(x)$ over L such that $\sigma W(D) = W(D')$. Let F be the affine model of $I[x]$. Since σ is an automorphism over L, it follows that $\sigma\sigma_i(A \otimes F) = \sigma'_i(A_i \otimes F)$, where σ_i and σ'_i are transition mappings of $W(D)$ and $W(D')$ respectively $(\sigma_i(x) = g_i x, \sigma'_i(x) = g'_i x)$. Let $\mathfrak{o}_i$ be the affine ring of A_i. Then we have $\mathfrak{o}_i[\sigma\sigma_i(x)] = \mathfrak{o}_i[\sigma'_i(x)]$, i.e., if a and b are elements of L such that $\sigma(x) = ax + b$, then $\mathfrak{o}_i[ag_i x + bg_i] = \mathfrak{o}_i[g'_i x]$. Then we have $g_i b \in \mathfrak{o}_i$ and ag_i/g'_i is a unit in $\mathfrak{o}_i$. Therefore ag_i is also a local equation of D' on A_i and $D' - D = (a)$. Thus D is linearly equivalent to D'. Now, let W be an arbitrary line bundle with base M and fibre F defined by $I[x]$. Let $A_1, \cdots, A_n$ and transition mappings $\sigma_1, \cdots, \sigma_n$ be as in the definition of line bundles. Since $\sigma_i(x) = g_i x$ with $g_i \in L$ and since g_i/g_j is a unit in every spot in $A_i \cap A_j$, there is a member of G_1 which has the g_i as local equations on the A_i, say D. Then $W = W(D)$. Thus the proof is completed.

Chapter 6. Simple Spots.

1. The set of simple spots in a model. A model M is called a ***non-singular model*** if every spot in M is a simple spot; it is called an ***unramified non-singular model*** if every spot in M is an unramified simple spot.

Remark 1. Though the notion of non-singular model does not depend on the choice of ground ring, the notion of unramified non-singular model depends on the choice of ground ring. If the ground ring is a field, then these two notions coincide to each other.

An affine model A is called a model of the *affine n-space* over a ground ring I if the affine ring of A is isomorphic to the polynomial ring in n algebraically independent elements over I (and if it is regarded as a model over I). A projective model M over I is called a model of the *projective n-space* if it is defined by a homogeneous coordinate system $(z_0, \cdots, z_n)$ with $n+1$ algebraically independent elements z_i. The dimension of these models

is either n or $n+1$ according to whether I is a field or not. A model of the projective n-space is the union of $n+1$ models of affine n-space.

Remark 2. Any affine (or projective) model can be regarded as an induced model of a model of affine (or projective) n-space for some n.

PROPOSITION 1. *If M is a model of affine or projective n-space, then M is an unramified non-singular absolutely irreducible model.*

Proof. This follows immediately from Corollary 5 to Proposition 1.1 and its proof.

LEMMA 1. *If a function field L over a ground ring I is separably generated over I, then any model M of L contains an unramified non-singular model (over I).*

Proof. We have only to show that there exists an unramified non-singular model of L; for if it is done, then the intersection of the model with M is a model by Theorem 2.10 and is an unramified non-singular model. Now, let $x_1, \cdots, x_n$ be a separating transcendence base of L over I and let $b \in L$ be such that $\mathfrak{o} = I[x_1, \cdots, x_n, b]$ is an affine ring of L and such that b is integral over $I[x_1, \cdots, x_n]$. Let d be the discriminant of the irreducible monic polynomial over $I[x_1, \cdots, x_n]$ which has b as a root. Set $\mathfrak{o}' = \mathfrak{o}[1/d]$. We shall show that the affine model A defined by $\mathfrak{o}'$ is an unramified non-singular model. Let P be any spot in A. Then P dominates a spot P^* in the affine model A^* defined by $I[x_1, \cdots, x_n]$. P^* is an unramified simple spot by Proposition 1. By our construction, P is a ring of quotients of $P^*[b]$ with respect to a maximal ideal. Since d is a unit in $\mathfrak{o}'$, d is a unit in P^* and therefore P is an unramified simple spot by Proposition 3.2. Thus A is an unramified non-singular model, which proves Lemma 1.

Remark 3. There are models over some ground ring, say I, which does not contain any unramified non-singular model over I.

Example. Let k be a field of characteristic $p \neq 0$ which contains infinitely many elements and let x be a transcendental element over k. Set $I = k[x^p]$ and $\mathfrak{o} = k[x]$. Then $\mathfrak{o}$ is an affine ring over I. I contains infinitely many prime ideals of the form $(x^p - a^p)I$ $(a \in k)$ and these prime ideals ramify in $\mathfrak{o}$. Therefore the affine model A defined by $\mathfrak{o}$ contains infinitely many ramified simple spots over I. Since $\dim A = 1$, this shows that A cannot contain any unramified non-singular model over I.

LEMMA 2. *Any model contains a non-singular model.*

Proof. By the same reason as in the proof of Lemma 1, we have only to prove that any function field L over a ground ring I has a non-singular model. Let p be the characteristic of I. If $p=0$, L is separably generated over I, and the assertion is true by Lemma 1. Therefore we assume that $p\neq 0$. Let $a_1,\cdots,a_n$ be elements of some integral extension of I such that 1) $a_i p\in I[a_1,\cdots,a_{i-1}]$ for every $i=1,\cdots,n$ and 2) $L(a_1,\cdots,a_n)$ is separably generated over $I[a_1,\cdots,a_n]$. We shall prove the assertion by induction on n. If $n=0$, then L is separably generated and the case was settled by Lemma 1. Assume that $n\geqq 1$ and let I^* be the derived normal ring of $I[a_1]$. Then by our induction assumption, there exists a non-singular model M^* of $L(a_1)$ over I^*. Since I^* is a finite I-module, we may regard M^* as a model over I. Let $\mathfrak{o}$ be an affine ring of L and set $\mathfrak{o}'=\mathfrak{o}[a_1]$. Let M and M' be the affine models defined by $\mathfrak{o}$ and $\mathfrak{o}'$ respectively. Then, since a_1 is purely inseparable, the projection from M' into M is a one-one and onto mapping; if $\operatorname{proj} P'=P$, then $P'=P[a_1]$. Since M^* is a non-singular model of $L(a_1)$, $M^*\cap M'$ is a non-singular model of $L(a_1)$. Therefore there exists an element f of $\mathfrak{o}'$ such that the model A' defined by $\mathfrak{o}'[1/f]$ is a non-singular model. By Corollary 1 to Proposition 4.1, every spot in $\operatorname{proj}_M A'$ is a simple spot. Since $\operatorname{proj} A'$ is obviously the affine model defined by $\mathfrak{o}[1/f^p]$, the assertion is proved.

PROPOSITION 2. *Let P be a simple spot in a model M and let $x_1,\cdots,x_r$ be a regular system of parameters of P. Then the induced model $\phi_P(M)$ contains a model M' such that if a spot $\phi_P(Q)$ $(Q\in M(P))$ is in M', then Q is a simple spot and has a regular system of parameters which contains $x_1,\cdots,x_r$ as a subset.*

Proof. We may assume without loss of generality that M is an affine model. Let $\mathfrak{o}$ be the affine ring of M and let $\mathfrak{p}$ be the prime ideal of $\mathfrak{o}$ such that $P=\mathfrak{o}_{\mathfrak{p}}$. Furthermore, let $\mathfrak{a}$ be the ideal generated by $x_1,\cdots,x_r$ in $\mathfrak{o}$. Since $\mathfrak{a}P=\mathfrak{p}P$, $\mathfrak{p}$ is a minimal prime divisor of $\mathfrak{a}$ and the primary component of $\mathfrak{a}$ belonging to $\mathfrak{p}$ is just $\mathfrak{p}$. Let f be an element of $\mathfrak{o}$ such that f is not in $\mathfrak{p}$ and is in every prime divisor of $\mathfrak{a}$ except for $\mathfrak{p}$. Then we may consider the affine model defined by $\mathfrak{o}[1/f]$ instead of M. Thus we may assume that $\mathfrak{a}=\mathfrak{p}$. Let M' be a non-singular model contained in $\phi_P(M)$; existence follows from Lemma 2. Let Q be any spot in $M(P)$ such that $\phi_P(Q)\in M'$. Then since $\mathfrak{a}=\mathfrak{p}$, $\phi_P(Q)=Q/\mathfrak{a}Q$. Therefore, that $\phi_P(Q)$ is simple shows that Q is also simple and that $x_1,\cdots,x_r$ are contained in a regular system of parameters of Q (Lemma 0.12). Thus the proof is completed.

Remark 4. In the above proposition, (i) if P is an unramified simple spot and if P dominates a ground place which is not a field, then any Q obtained by the proof above is also an unramified simple spot, for a prime element of the ground place dominated by P can be a member of a regular system of parameters of P; (ii) if P contains a field of quotients of I and if M' is an unramified non-singular model over I, then Q is an unramified simple spot.

By virtue of Theorem 2.9, Proposition 2.6 can be stated as follows:

PROPOSITION 3. *Let M be a model. Then a subset M' of M is a model if and only if the following conditions are satisfied:*

1) *M' is not empty.* 2) *If a spot $P \in M$ is not in M', then any specialization of P in M is not in M'.* 3) *If a spot $P \in M$ is in M', then $\phi_P(M(P) \cap M')$ contains a model.*

THEOREM 1. *The set M' of simple spots in a model M is also a model.*

Proof. We shall make use of Proposition 3. The condition 1) is obvious. Thetorem 4.1 shows the validity of the condition 2). Proposition 2 shows the validity of 3). Thus M' is a model.

PROPOSITION 4. *Let M be a model of a function field L over a ground ring I. Then the set M' of unramified simple spots in M is a model (over I) if at least one of the following conditions are satisfied:* 1) *I is a semi-local ring (or a field).* 2) *I is of characteristic zero.*

Proof. M' is obviously non-empty. Theorem 4.2 shows that if $P \in M$ is not in M', then any specialization of P is not in M'. Assume that $P \in M'$. If P dominates a ground place I' which is not a field, then Proposition 2 and Remark 4 shows the validity of Condition 3) in Proposition 3. Assume that P contains the field of quotients k of I. i) In the case 1), the set of spots in $\phi_P(M)$ which contains k contains a model and therefore $\phi_P(M(P) \cap M')$ contains a model by Remark 4. ii) In the case 2), since $\phi_P(P)$ is separably generated over I, $\phi_P(M(P) \cap M')$ contains a model by Lemma 1 and Remark 4. Therefore, by Proposition 3, we prove Proposition 4.

2. Criteria for unramified simple spots. Let $X_1, \cdots, X_n$ be algebraically independent elements over an integral domain I and let $f_1, \cdots, f_r$ be elements of $I[X_1, \cdots, X_n]$. Then the Jacobian matrix $(\partial f_i/\partial X_j)$ will be denoted by $J(f_1, \cdots, f_r)$. Let I^* be a subring of I. Then the set $\mathfrak{D}_{I/I^*}$ of integral derivations of I over I^* is an I-module. Let $\{D_\sigma\}$ be a set of

generators of $\mathfrak{D}_{I/I^*}$. Then the matrix $(\partial f_i/\partial X_j; f_i^{D\sigma})$ is called a *mixed Jacobian matrix* of $f_1,\cdots,f_r$ with respect to I^* and is denoted by $J^*(f_1,\cdots,f_r;I^*)$. Observe that $J^*(f_1,\cdots,f_r;I^*)$ is substantially unique, i.e., change of the generators of $\mathfrak{D}_{I/I^*}$ corresponds to a linear transformation of columns. Observe also that $J(f_1,\cdots,f_r)=J^*(f_1,\cdots,f_r;I)$.

THEOREM 2. *Assume that $I=k$ is a field and set $A=k[X_1,\cdots,X_n]$. Let $\mathfrak{p}\subseteq\mathfrak{q}$ be prime ideals of A and set $R=A_{\mathfrak{q}}$. Furthermore, let $\mathfrak{a}=\sum_1^r f_iA$ be an ideal having $\mathfrak{p}$ as a prime divisor. Then:*

(1) *If $A/\mathfrak{q}$ is separably generated over k, $R/\mathfrak{a}R$ is a simple spot if and only if rank $(J(f_1,\cdots,f_r)$ modulo $\mathfrak{q})=$ rank $\mathfrak{p}$.*

(2) *If k is of characteristic $p\neq 0$, then $R/\mathfrak{a}R$ is a simple spot if and only if there exists a subfield k^* of k such that $[k:k^*]$ is finite and such that rank $(J^*(f_1,\cdots,f_r;k^*)$ modulo $\mathfrak{q})=$ rank $\mathfrak{p}$; or equivalently, rank$(J^*(f_1,\cdots,f_r;k^p)$ modulo $\mathfrak{q})=$ rank $\mathfrak{p}$.*

For the proof, see Nagata [4] or Zariski [8].

It should be remarked here that when $\mathfrak{p}$ is a prime ideal of $I[X_1,\cdots,X_n]$, $\mathfrak{p}=\sum_1^r f_iI[X_1,\cdots,X_n]$, that $\operatorname{rank}(J(f_1,\cdots,f_r))=\operatorname{rank}\mathfrak{p}$ means that 1) $\mathfrak{p}\cap I=0$ and 2) $I[X_1,\cdots,X_n]/\mathfrak{p}$ is separably generated over I.

As for the case of spots over a ground ring, we have the following application of Theorem 2:

Assume that I is a ground ring, $\mathfrak{o}=I[X_1,\cdots,X_n]$, $\mathfrak{p}\subseteq\mathfrak{q}$ are prime ideals of $\mathfrak{o}$, $\mathfrak{a}=\sum_1^r f_i\mathfrak{o}$ is an ideal having $\mathfrak{p}$ as a prime divisor and that $\mathfrak{p}\cap I=0$. Set $R=\mathfrak{o}_{\mathfrak{q}}$. Then

THEOREM 3. *If rank $(J(f_1,\cdots,f_r)$ modulo $\mathfrak{q})=$ rank $\mathfrak{p}$, then $R/\mathfrak{a}R$ is an unramified simple spot. Conversely, if $R/\mathfrak{a}R$ is a tamely unramified simple spot, then rank $(J(f_1,\cdots,f_r)$ modulo $\mathfrak{q})=$* rank $\mathfrak{p}$.

Proof. If the field of quotients of I is contained in $R/\mathfrak{a}R$ (*i.e.*, $\mathfrak{q}\cap I=0$), then the assertion is contained in Theorem 2. Therefore we assume that I is not a field and is a ground place dominated by R. Let p be a prime element of I ($p\in\mathfrak{q}$). If rank $(J(f_1,\cdots,f_r)$ modulo $\mathfrak{q})=$ rank $\mathfrak{p}$, then we have $R/(pR+\mathfrak{a}R)$ is a simple spot over I/pI. Therefore $(pR+\mathfrak{a}R)/\mathfrak{a}R\supseteq\mathfrak{p}R/\mathfrak{a}R$, and $\mathfrak{p}R=\mathfrak{a}R$ and therefore $R/\mathfrak{a}R$ is an unramified simple spot over I. Conversely, if $R/\mathfrak{a}R$ is a tamely unramified simple spot, then $R/(pR+\mathfrak{a}R)$ is a tamely unramified simple spot over I/pI, hence rank $(J(f_1,\cdots,f_r)$ modulo $\mathfrak{q})=$ rank $\mathfrak{p}$.

COROLLARY. *If a model M over a ground ring I carries a tamely unramified simple spot, then the function field of M is separably generated over I.*

Remark. The second criterion in Theorem 2 can be applied for unramified simple spots. But it depends on the ground place dominated by the spot (apply the criterion in the case where k is the residue class field of the ground place).

We shall give here a remark on projective models.

Let M be a projective model of a function field L over a ground ring I and let $\mathfrak{H}=I[z_0,\cdots,z_n]$ be a homogeneous coordinate ring which defines M. Then $\mathfrak{H}$ is an affine ring of $L(z_i)$ with some $z_i\neq 0$ and such z_i is a transcendental element over L.

Now let P be a spot in M. Then there exists a uniquely determined prime ideal $\mathfrak{p}$ of $\mathfrak{H}$ such that if $z_i\notin\mathfrak{p}$ then a homogeneous form $f\in\mathfrak{H}$ is in $\mathfrak{p}$ if and only if f/z_i^d $(d=\deg f)$ is in the maximal ideal of P and $\mathfrak{H}/\mathfrak{p}$ is a homogeneous coordinate ring of the induced model $\phi_P(M)$. Then obviously $\mathfrak{H}_\mathfrak{p}$ dominates P. Since $\mathfrak{H}/\mathfrak{p}$ is a homogeneous coordinate ring of $\phi_P(M)$, we see that $\mathfrak{H}_\mathfrak{p}=P(z_i)$ with some $z_i\notin\mathfrak{p}$.

By virtue of this fact, it is sometimes convenient to consider the affine model M^* defined by $\mathfrak{H}$, which is called the model of a representative cone of M. For example, P is a simple spot (or a normal spot) if and only if the corresponding spot $\mathfrak{H}_\mathfrak{p}$ is simple (or normal, respectively). Thus, we can apply the Jacobian criterion for simple spots in M, using the homogeneous coordinate ring $\mathfrak{H}$, namely:

Let $\mathfrak{o}$ be the polynomial ring in indeterminates $Z_0,\cdots,Z_n$ over I and let $\mathfrak{P}$ be the kernel of the homomorphism ϕ from $\mathfrak{o}$ onto $\mathfrak{H}$ such that $\phi(Z_i)=z_i$. Let $f_1,\cdots,f_r$ be a set of generators of $\mathfrak{P}$. Then we get a criterion for simplicity of spots in M, quite similar to the assertion in Theorem 3, using $J(f_1,\cdots,f_r)$.

3. Absolutely simple spots. A spot P is called *absolutely simple* if 1) P is a regular extension of the ground ring I and 2) for any ground ring I^* containing I, any extension of P over I^* is a simple spot.

A model M over a ground ring I is said to be *absolutely non-singular* if every spot in M is absolutely simple. For example, a model of the affine (or projective) n-space is absolutely non-singular.

THEOREM 4. *An absolutely simple spot is an unramified simple spot.*

Proof. Let P be an absolutely simple spot over a ground ring I. Assume

that P is ramified. We may assume that I is a valuation ring with a prime element x. Then x is in the square of the maximal ideal $\mathfrak{m}$ of P. Let $y_1,\cdots,y_r$ be a regular system of parameters of P. Set $I'=I[a]$ with $a^2=x$. Then I' is a valuation ring. The extension P^* of P over I' is $P[a]$. Since, by assumption, $P[a]$ is a simple spot, and since the maximal ideal $\mathfrak{m}^*$ of P^* is generated by $a, y_1,\cdots,y_r$, there is a non-trivial relation $ac_0+\sum_1^r y_ic_i = a\sum_1^r y_id_i+\sum_{ij}y_iy_je_{ij}$ with $c_i, d_i, e_{ij}\in P$ and $c_j=0$ if $c_j\in\mathfrak{m}$. Since 1, a are linearly independent over P, we have $c_0=\sum y_id_i$, $\sum y_ic_i\in\mathfrak{m}^2$, hence all c_i must be zero, which is a contradiction. Thus we prove Theorem 4.

THEOREM 5. *With the same notations as in Theorem* 3, $R/\mathfrak{a}R$ *is an absolutely simple spot if and only if* 1) *it is a regular extension of* I *and* 2) *rank* $(J(f_1,\cdots,f_r)$ *modulo* $\mathfrak{q})=$ *rank* $\mathfrak{p}$.

Proof. The conditions are independent of ground ring extensions. Therefore Theorem 5 follows easily from Theorems 3 and 4.

COROLLARY 1. *Let* M *be an absolutely irreducible model over a ground ring* I. *Then the set* M' *of absolutely simple spots forms a model* (*over* I).

Proof. Let A be an arbitrary affine model contained in M and let $\mathfrak{S}$ be the affine ring of A. Then there exists a polynomial ring $\mathfrak{o}=I[X_1,\cdots,X_n]$ such that $\mathfrak{S}=\mathfrak{o}/\mathfrak{p}$ with a prime ideal $\mathfrak{p}$ of $\mathfrak{o}$. Let $f_1,\cdots,f_r$ be a set of generators of $\mathfrak{p}$ and let $d_1,\cdots,d_s$ be the set of subdeterminants of degree $=\operatorname{rank}\mathfrak{p}$, of the Jacobian matrix $J(f_1,\cdots,f_r)$. Let $\mathfrak{b}$ be the ideal of $\mathfrak{S}$ generated by the residue classes of the d_i modulo $\mathfrak{p}$. Then Theorem 5 shows that $A\cap(M-M')$ is a closed set of A defined by the ideal $\mathfrak{b}$. Therefore $A\cap(M-M')$ is a closed set for any affine model A contained in M, which shows that $M-M'$ is a closed set by Theorem 2.8, hence M' is a open set of M. Since M' is not empty (for example, the function field L of M is in M'), we see that M' is a model over I by Theorem 2.9.

COROLLARY 2. *A tamely unramified simple spot over a ground ring* I *is an absolutely simple spot if it is a regular extension of* I.

Remark. Zariski [8] defined absolutely simplicity without the assumption of regularity of the extension: In his sense, a spot P over a ground field k is absolutely simple if for any extension field k^* of k and for any prime ideal $\mathfrak{n}$ of $P\otimes k^*$ containing the maximal ideal of P, $(P\otimes k^*)_{\mathfrak{n}}$ is a simple spot over k^* (the condition is equivalent even if we restrict k^* to algebraic extensions of k). Under this definition, absolute simplicity is characterized by the condition 2) in Theorem 5. But, if the same definition is applied to

spots over ground rings, then as is easily seen, Theorem 4 becomes false (for instance, let I' be a valuation ring which is finite separable and ramified extension of a ground place I and let P be an absolutely simple spot in our sense which dominates I'. If we regard P as a spot over I, then P becomes an absolutely simple spot in the sense of Zariski and is ramified over I). Therefore the condition 2) in Theorem 5 is not a necessary condition for absolute simplicity in this weak sense.

4. The set of absolutely normal spots.

THEOREM 6. *Let M be an absolutely irreducible model over a ground ring I. Then the set M' of absolutely normal spots in M forms a model over I.*

Proof. 1) When I is a field, let I^* be the smallest perfect field containing I. Then Theorem 5.9 shows that the set of spots in $M \otimes I^*$ which are absolutely normal is the set $M' \otimes I^*$ of spots in $M \otimes I^*$ which are extensions of spots in M'. Since I^* is perfect, $M' \otimes I^*$ is nothing but the set of normal spots in $M \otimes I^*$ by Corollary 2 to Theorem 5.6, hence it is a model. Now, observing the natural one-one correspondence between spots in M and $M \otimes I^*$, we see easily that M' is an open set of M, hence M' is a model because it is non-empty.

2) Now we consider the case where I is not a field. Let k be the field of quotients of I. Lemma 5.4.1 shows that $P \in M$ is absolutely normal if and only if i) any spot which is a ring of quotients of $P[k]$ is absolutely normal and ii) if $P(\mathfrak{p})$ is a general spot over a prime ideal $\mathfrak{p}$ of I and if P is a specialization of $P(\mathfrak{p})$ then $P(\mathfrak{p})$ is absolutely normal. The absolute normality of $P(\mathfrak{p})$ is equivalent to the absolute simplicity of $P(\mathfrak{p})$. Since the set of absolutely simple spots in M forms a model by Corollary 1 to Theorem 5, there exists only a finite number of general spots over some prime ideals of I which are not absolutely simple: Let them be $P_1, \cdots, P_n$. Then M' is contained in $M - (\cup_i M(P_i))$. Therefore, considering $M - (\cup_i M(P_i))$, we may assume that all the $P(\mathfrak{p})$ are absolutely simple. Let M^* be the restricted model of M over k. Then by 1), the complement of $M^* \cap M'$ in M^* is a closed set of M^*, hence there are a finite number of spots $Q_1, \cdots, Q_m$ in M^* such that the closed set if the union of the $M^*(Q_i)$. Now, Lemma 5.4.1, quoted above, shows that $M' = M - (\cup_i M(Q_i))$, which proves Theorem 6.

5. Simple points. We shall use the same notations as in § 5, Chapter 5. Then

THEOREM 7. *The following three conditions are equivalent to each other:*

(1) *The point P in the variety M' is a simple point.*

(2) *Every P_σ is absolutely simple.*

(3) *There exists one I_σ such that P_σ is absolutely simple.*

Proof. Assume that (1) is true. Then considering I_σ such that $\phi_P(P_\sigma)$ is separably generated over $\phi_P(I_\sigma)$, we see that (3) is true by Theorems 3 and 5. Since the Jacobian matrix is independent of I_σ, we see that (2) follows from (3). It is obvious that (1) follows from (2). Therefore Theorem 7 is proved.

We remark that Corollary 1 to Theorem 5 and Theorem 6 show respectively that the set of simple points and the set of normal points form open sets of the variety M' in I-topology.

6. Tensor products of simple spots.

THEOREM 8. *Let P and P' be simple spots which dominates the same ground place I.*

(1) *If $\phi_P(P)\otimes_{\phi_P(I)}\phi_{P'}(P')$ has no nilpotent element (other than zero) and if P is unramified, then for any prime ideal $\mathfrak{p}$ of $P\times_I P'$, $(P\times P')_\mathfrak{p}$ is a simple spot over I.*

(2) *If P and P' are unramified, then for any prime ideal $\mathfrak{p}$ of $P\otimes_I P'$, $(P\otimes P')_\mathfrak{p}$ is an unramified simple spot over I.*

Proof. (1) Let $\mathfrak{m}$ and $\mathfrak{m}'$ be the maximal ideals of P and P' respectively. By our assumption, the ideal $\mathfrak{a}$ generated by $\mathfrak{m}$ and $\mathfrak{m}'$ in $P\times P'$ is semi-prime. If $\mathfrak{n}$ is a maximal ideal of $P\times P'$, then $\mathfrak{n}(P\times P')_\mathfrak{n}$ is generated by $\mathfrak{a}$. i) When I is a field, rank $(P\times P')_\mathfrak{n}=\operatorname{rank} P+\operatorname{rank} P'$ and we see that $(P\times P')_\mathfrak{n}$ is a simple spot. ii) If I is a valuation ring having a prime element x, considering a regular system of parameters of P which has x as a member, we see that $(P\times P')_\mathfrak{n}$ is a simple spot. Now we see that $(P\times P')_\mathfrak{p}$ is a simple spot by Theorem 4.1.

(2) If I is a field, we see the validity of (2) by the criterion of simplicity by mixed Jacobian matrix as follows: Let $I[X_1,\cdots,X_m]$ and $I[Y_1,\cdots,Y_n]$ be polynomial rings in indeterminates X_i and Y_j such that $P=I[X_1,\cdots,X_m]_{\mathfrak{M}}/(f_1,\cdots,f_r)$, $P'=I[Y_1,\cdots,Y_n]_{\mathfrak{M}'}/(g_1,\cdots,g_s)$ with prime ideals $\mathfrak{M}$ and $\mathfrak{M}'$ and elements f_i and g_j. Let $\mathfrak{P}$ be the prime ideal of $I[X_1,\cdots,X_m,Y_1,\cdots,Y_n]$ such that

$$(P\otimes P')_{\mathfrak{p}}=I[X_1,\cdots,X_m,Y_1,\cdots,Y_n]_{\mathfrak{P}}/(f_1,\cdots,f_r,g_1,\cdots,g_s).$$

Since rank $(J^*(f_1,\cdots,f_s;I^p)$ modulo $\mathfrak{M})=$ rank $(f_1,\cdots,f_r)$ and

$$\text{rank }(J^*(g_1,\cdots,g_s;I^p)\text{ modulo }\mathfrak{M}')=\text{rank }(g_1,\cdots,g_s),$$

we have

$$\text{rank }(J^*(f_1,\cdots,f_r,g_1,\cdots,g_s;I^p)\text{ modulo }\mathfrak{P})$$
$$=\text{rank }(f_1,\cdots,f_r,g_1,\cdots,g_s),$$

which proves the simplicity of $(P\otimes P')_{\mathfrak{p}}$ (since I is a field, it is unramified). If I is a valuation ring, let x be a prime element of I. Then by the case where I is a field, $(P\otimes P')_{\mathfrak{p}}/(x)$ is a simple spot over I/xI, hence $(P\otimes P')_{\mathfrak{p}}$ is an unramified simple spot.

COROLLARY. *If M and M' are absolutely non-singular models over the same ground ring I, then $M\otimes M'$ is also absolutely non-singular and conversely. If M and M' are unramified non-singular models over I and if $M\otimes M'$ is well defined, then $M\otimes M'$ is also unramified non-singular and conversely.*

Proof. The last half follows from Theorem 8 and its proof, while the first half can be proved by the same way using Jacobian matrices.

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ON THE TRANSFORMATION THEORY OF ELLIPTIC FUNCTIONS.*[1]

By Jun-ichi Igusa.

It has been customary to call the *theory of isogeny of ellipitic curves with "variable moduli"* as transformation theory of elliptic functions. The results of Lagrange and Gauss on arithmetic-geometric means and transformations discovered by Landen and Legendre were the germs of this theory. However, a real initiation was made independently by Abel and Jacobi. This theory was later deepened and completed in an essential way by Kronecker in his well-known series of papers on elliptic functions. Now, in a 1950 congress address, André Weil gave a geometric formulation of the transformation theory and suggested a reconsideration of this "splendid work" from a modern point of view. However, as far as we know, no one has yet worked out his suggestion. This we shall propose to do in this paper. As we know, although the final results are of geometric nature, Kronecker made use of Jacobi elliptic functions and, what is worse, the transformation formulas of Jacobi theta functions in establishing his theory. Now, it seems that any worth-while reconsideration should be along the *strict Kroneckerian programme* and, in particular, it should be of a geometric nature. It is relatively easy to discuss contributions of Abel and Jacobi in a geometric form. However, we encountered with one difficulty in geometrizing Kronecker's theory. In fact, we are thus led to establish a new geometric theory of modular functions, which we shall publish separately. With the aid of this theory, we shall discuss the transformation theory in a *definitive form* which is more precise than Weil's formulation in various respects. We hope that this paper contributes to the clarification of the matter and thus to a future generalization of the theory.

1. Preliminaries. Throughout this paper, we shall assume that the

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characteristics of fields we consider are different from 2. Moreover, by an elliptic curve we always understand an Abelian variety of dimension 1. We shall denote its neutral element by o. In this section we shall summarize basic results on elliptic curves, which hold actually for any characteristic. Most of them are proved by Hasse in [3]. However, to readers with geometric background, we shall recommend Weil's general treatment [9]. We shall make use of some more results, which we shall mention at suitable places.

Consider the group of divisors of an elliptic curve A. Then the natural epimorphism of this group to the group variety A will be denoted by S. Let $\mathfrak{a}$ be a divisor of A of degree zero. Then $\mathfrak{a}$ belongs to the kernel of S if and only if $\mathfrak{a}$ is a principal divisor, i.e. if and only if there exists a function f on A satisfying $(f)=\mathfrak{a}$. In this way S gives rise to an isomorphism of the divisor class-group of degree zero of A, i.e. the group of divisors of A of degree zero modulo the group of principal divisors of A, to A. We can simply say that A is its own Albanese and Picard variety. On the other hand, if α is a homomorphism of A to another elliptic curve B, then $S\cdot\alpha^{-1}$ is a homomorphism α' of B to A up to a translation in A. The correspondence $\alpha\rightarrow\alpha'$ is an isomorphism of $\mathrm{Hom}(A,B)$ to $\mathrm{Hom}(B,A)$. Moreover $\alpha\alpha'$ is $\deg(\alpha)$ in the sense $\alpha\alpha'(v)=\deg(\alpha)v$ for every point v of B. Here $\deg(\alpha)$ means the mapping degree of α. In a similar sense, we have $\alpha'\alpha=\deg(\alpha)$. If β is a homomorphism of B to another elliptic curve C, we have $(\beta\alpha)'=\alpha'\beta'$ and $(\alpha')'=\alpha$. There is another way to associate a contravariant map to α. In fact, if we consider the vector spaces of invariant differentials of A and B, then α induces a contravariant linear map of these vector spaces. Therefore, if we fix their bases θ_A and θ_B, we get a finite quantity μ by $\theta_B\rightarrow\mu\theta_A$. The quantity μ is called the *multiplicator* of α. The multiplicator of α is zero if and only if α is an inseparable homomorphism. Moreover, in case $A=B$, we can take $\theta_A=\theta_B$. Then the multiplicator of α is uniquely determined by α and the correspondence $\alpha\rightarrow\mu$ is a representation of the ring $\mathrm{Hom}(A,A)$. We note that an invariant differential and an everywhere finite differential are the same.

If n is a natural number, the endomorphism n of A is of degree n^2. This endomorphism is separable if and only if n is not a multiple of the characteristic. In this case, therefore, the kernel is a finite Abelian group of type (n,n). On the other hand, if $\mathfrak{a}$ is a divisor of A of degree zero such that $t=S(\mathfrak{a})$ is a point of order n, we have $S(n^{-1}(\mathfrak{a}))=o$. Hence, there exists a function f on A satisfying $(f)=n^{-1}(\mathfrak{a})$. If s is a point of A of order n and u a general point of A, we have $f(u+s)=e_n(s,t)f(u)$ with some n-th root of unity $e_n(s,t)$. This n-th root of unity depends only on n,

s and t. Moreover, the correspondence $(s,t) \rightarrow e_n(s,t)$ is a skew-symmetric pairing of the group of points of A of order n to itself. This fact will play a role immediately in the next section.

There is one more theorem, which is proved by Chow [1]. Let α be, as above, an element of $\mathrm{Hom}(A,B)$. Then α is always defined over a separable extension of a common field of definition of A and B. Actually, as most of the other results, it is proved for general Abelian varieties.

2. Jacobi quartics. Let A be an elliptic curve. Then A carries sixteen points of order four. Pick one, say r, out of twelve primitive fourth division points, i.e. points of exact order four. Then, there exists a function x on A determined up to a constant by $(x) = 2^{-1}(0) - 2^{-1}(2r)$. Since (x) is invariant under the automorphism $u \rightarrow -u$ of A, we have $x(-u) = cx(u)$ with some constant c different from 0. Since the automorphism is of order 2, we get $c^2 = 1$. However, since $u = 0$ is a simple zero of x, as we can see by a specialization argument, it can not be an even function. Therefore x must be an odd function, i.e. we have $x(-u) = -x(u)$. On the other hand, (x) will be transformed into $-(x)$ under the translation $u \rightarrow u + r$ of A. Hence, we have $x(u+r) = cx(u)^{-1}$ with some constant c different from 0. *We shall normalize the constant factor of x so that we get $c = 1$. Then x is uniquely determined up to its sign and we have $x(u+r) = x(u)^{-1}$.* In particular, x is invariant under the translation $u \rightarrow u + 2r$ of A. Naturally, this property does not depend on the normalization. Furthermore, x undergoes sign changes by two other translations of A of order 2.

Now, let s be another primitive fourth division point of A such that $2s$ is different from $2r$. We have eight possible s and the mean value ρ of x^2 over these eight points

$$\rho = (\tfrac{1}{8}) \sum x^2(s)$$

is uniquely determined by A and $\pm r$. Here we remark this. If σ is one of the $x(s)$, the eight values of $x(s)$ are simply $\pm\sigma$, $\pm\sigma^{-1}$ each with multiplicity 2. Therefore ρ is of the form $\frac{1}{2}(\sigma^2 + \sigma^{-2})$ and we have the following relation

$$\prod (x(s) - X) = (1 - 2\rho X^2 + X^4)^2.$$

We also make the following remark. If v is a point of A at which x is finite, then, including multiplicity, the zeros of $x - x(v)$ are given modulo $2r$ by v and $-v + 2s$. As an application of this fact, we can determine the divisor of the function $1 - 2\rho x^2 + x^4$ on A and we get $2\mathfrak{a}$ with $\mathfrak{a} = \sum(s) - 2 \cdot 2^{-1}(2r)$.

Since we have $S(a)=0$, we can write $1-2\rho x^2+x^4$ in the form y^2 with some function y on A. Since (y) is invariant under the automorphism $u\to -u$ of A, we see that y is even or odd. However, sinve we have $y^2(0)=1$, it must be an even function, i. e. we have $y(-u)=y(u)$. *We shall normalize y uniquely by $y(0)=1$.* The two functions x and y are invariant under the translation $u\to u+2r$ of A and "uniformize" the plane quartic $Y^2=1-2\rho X^2+X^4$, which we call the *Jacobi quartic* of *modulus* ρ[2] We note that the modulus ρ is different from ± 1 and ∞. It is clear that ρ is finite. If ρ is ± 1, we get $y=1\mp x^2$. However, this is not possible, because y must undergo sign changes by the two translations of A of the form $u\to u+2s$.

Conversely, if ρ is different from ± 1 and ∞, the Jacobi quartic of modulus ρ is absolutely irreducible and the four roots $\pm\sigma$, $\pm\sigma^{-1}$ of $1-2\rho X^2+X^4=0$ are distinct. Moreover, the point at infinity $(0,1,0)$ is the only singularity. Therefore, the quartic is of genus 1 and, if Q is the prime field, we can introduce a normal law of composition over the field $Q(\rho)$ with $(0,1,1)$ as neutral element. *As long as we avoid the double point* $(0,1,0)$, i. e., *as long as we restrict to finite points, we can treat the Jacobi quartic as an elliptic curve.* In fact, the normalization over $Q(\rho)$ transforms the quartic into an elliptic curve and the correspondence is everywhere biregular except at the point at infinity. The *absolute invariant* of this elliptic curve can be calculated, for instance, by transforming it isomorphically into a plane cubic $Y^2=X(1-X)(\lambda-X)$ with $(0,1,0)$ as neutral element. The modulus λ is a cross-ratio of the four roots $\pm\sigma$, $\pm\sigma^{-1}$, hence $\lambda=\frac{1}{2}(1+\rho)$ is one of them. This shows that ρ is also a modular function of level 2 [4]. Moreover, in terms of ρ the absolute invariant can be expressed as follows

$$j=2^6(3+\rho^2)^3(1-\rho^2)^{-2}.$$

Therefore, *if Z is the prime integral domain, then 8ρ is always integral over $Z[j]$ and 8 is the smallest natural number having this property.* Moreover, the ρ-space is a Galois covering of the j-space with the symmetric group of permutations of three letters as the Galois group. The covering is ramified at the degenerate case $j=\infty$, the harmonic case $j=12^3$ and the equianharmonic case $j=0$. The corresponding values of ρ are given, respectively, by $\{\pm 1,\infty\}$, $\{0,\pm 3\}$ and $\{\pm(-3)^{\frac{1}{2}}\}$.

On the other hand, if a Jacobi quartic with a non-degenerate modulus ρ,

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[2] In the classical case, this quartic was uniformized by Jacobi elliptic functions as $x=\sigma\,\mathrm{sinam}(t,\sigma^2)$ and $y=\mathrm{cosam}(t,\sigma^2)\Delta\,\mathrm{am}(t,\sigma^2)$. See Jacobi [6, 7] and Kronecker [8].

i. e. a modulus ρ different from ± 1 and ∞, is given, we can always uniformize it by an elliptic curve A as indicated before. In fact, if we map the normalization of the quartic to an elliptic curve A by a homomorphism α of degree 2, essentially α' gives such a uniformization. We leave to the reader to make it precise. In this way, we can consider either the elliptic curve or the Jacobi quartic as initially given. At any rate, we shall always consider them as a pair. We also make the following rather scattered remarks.

If a_1 and a_2 are points of A of odd orders, then $x(a_1) = x(a_2)$ implies $a_1 = a_2$. If u is a variable point of A, then $x(u+s)$ is an even function of u. Also, the differential dx/y is everywhere finite on A, hence it is an invariant differential of A. Proofs are immediate. In the following sections, notations will be always the same. For instance, s means a primitive fourth division point of A such that $2s$ is different from $2r$.

3. Transformation of Jacobi quartics. Let A be an elliptic curve and let α be a homomorphism of an *odd* degree m of A to another elliptic curve A'. Then α induces an isomorphism of the 2-primary parts of the groups of points of finite orders of A and A'. In particular, if r is, as before, a primitive fourth division point of A, then $r' = \alpha r$ is a primitive fourth division point of A'. Let ρ and ρ' be the moduli of Jacobi quartics associated with A and A' with reference to $\pm r$ and $\pm r'$. Also, let x, y and x', y' be the functions which uniformize these quartics. Then α commutes with the translations $u \to u + 2r$ and $u' \to u' + 2r'$ of A and A', hence α induces a homomorphism of Jacobi quartics of the same degree m as α. Therefore, we can write $x'(\alpha u)$ uniquely in the form $R_0(x) + R_1(x)y$ with $x = x(u)$ and $y = y(u)$. Here $R_0(X)$ and $R_1(X)$ are rational functions of X. If we apply the automorphism $u \to -u$ of A, we see that R_0 and R_1 are both odd. If we apply the translation $u \to u + 2s$ of A, we see that R_1 is even. Hence R_1 must be zero and, thus, we simply get $x'(\alpha u) = R(x)$ with an odd rational function $R(X)$ of X. In the same way, we get $y'(\alpha u) = S(x)y$ with some even rational functions $S(X)$ of X. Since we know that the homomorphism $(x, y) \to (x', y')$ is of degree m, the rational function $R(X)$ must be of degree m. In other words, $R(X)$ is a quotient of co-prime polynomials at least one of which is of degree m. Now, if we apply the translation $u \to u + r$ of A, we get $R(X^{-1}) = R(X)^{-1}$. Since we also have $R(0) = 0$, we can write $R(X)$ in the form $cX^mF(X^{-1})F(X)^{-1}$ with $c^2 = 1$ and with

$$F(X) = 1 + \sum_{0 \leqq 2i < m-1} c_i X^{m-2i-1}.$$

Here, by changing the sign of x' if necessary, we can make $c = 1$. Remember

that x' is determined by r' only up to sign. If we substitute $x' = R(x)$ and $y' = S(x)y$ in $(y')^2 = 1 - 2\rho'(x')^2 + (x')^4$, we see that

$$F(X)^4 S(X)^2 (1 - 2\rho X^2 + X^4)$$

is a polynomial of X. Therefore $G(X) = F(X)^2 S(X)$ must be a polynomial of X and, in fact, a polynomial of X^2. Moreover, $G(X)$ is of degree $2m-2$ as a polynomial of X and satisfies $G(0) = 1$. With the aid of the polynomials $F(X)$ and $G(X)$, which we call transformation polynomials, the homomorphism $(x, y) \to (x', y')$ can be expressed as follows

$$x' = x^m F(x^{-1}) F(x)^{-1} \qquad y' = G(x) F(x)^{-2} y.$$

In Kronecker's terminology, *transformation equations* meant always $F(X) = 0$, or rather $X^m F(X^{-1}) = 0$ and mostly in the case of prime transformation degrees. We shall show that the transformation polynomials can be decomposed into linear factors explicitly.

Take a field of definition K of α containing all coefficients of $F(X)$. Let u be a generic point of A over K and put $x' = x'(\alpha u)$. Then x' is transcendental over K, hence $X^m F(X^{-1}) - x' F(X)$ is an irreducible polynomial over $K(x')$. It is easy to determine all distinct roots of this polynomial. At any rate, if a belongs to the kernel of α, certainly $x(u+a)$ is a root. Conversely, if x is an arbitrary root, we can write it in the form $x(v)$ with some point v of A and we get $x'(\alpha v) = x'$. In Section 2, we remarked how to analyse an equation of this type. Thus we get $x(v) = x(u+a)$ with some a in the kernel of α. Also, if a_1 and a_2 are distinct elements of the kernel of α, then $x(a_1)$ and $x(a_2)$, hence also $x(u+a_1)$ and $x(u+a_2)$, are distinct. Therefore, if e is the common multiplicity of the roots and if we denote a typical element of the kernel of α by a, we get

$$X^m F(X^{-1}) - x' F(X) = \prod_a (X - x(u+a))^e.$$

In particular, if we specialize u to 0 over K, we get

$$F(X) = \prod_a (1 - x(a) X)^e.$$

The exponent e is a certain power of the characteristic, and it is called the inseparability degree of α. The number of elements in the kernel of α multiplied by the inseparability degree e of α is the total degree m of α.

Now, in order to obtain a similar decomposition of $G(X)$ into linear factors, we shall explain the notion of a "half system." A half system of a finite Abelian group of an odd order is its subset S which together with

zero and $-S$ sum, without overlapping, to the whole group. We see that a point v of A satisfies $y'(\alpha v)=0$ if and only if v is of the form $s+a$, i.e., if and only if we have $x(v)=x(s+a)$ with some s and a. Here, if we restrict s to representatives modulo $2r$ and a either to 0 or to elements of a half system of the kernel of α, the corresponding $x(s+a)$ are distinct. Since $x(s+u)$ is an even function of u, we do not loose anything by this restriction. Moreover, certainly $x(s+a)$ are different from the roots of $F(X)=0$. Therefore, by using the previous remark, we get

$$G(X)=(1-2\rho X^2+X^4)^{\frac{1}{2}(e-1)}\prod_{\substack{s \bmod 2r\\ a \text{ half system}}}(X-x(s+a))^e.$$

On the other hand, with respect to the invariant differentials dx/y and dx'/y' we can calculate the multiplicator μ of α. In fact, if we observe $y(0)=y'(0)=1$, we get

$$\mu=c_0=\prod_{a\neq 0}x^e(a).$$

In the above discussion, we started from a homomorphism α of an odd degree of elliptic curves A and A' and we studied the induced homomorphism of Jacobi quartics associated with A and A' with reference to $\pm r$ and $\pm\alpha r$. Conversely, suppose that a non-degenerate Jacobi quartic of modulus ρ and a homomorphism $(x,y)\rightarrow(x'',y'')$ of an odd degree m of this quartic to another Jacobi quartic of modulus ρ'' are given. Then, at any rate, we can uniformize the Jacobi quartic of modulus ρ by an elliptics curve A with reference to, say $\pm r$, and we can consider the kernel of the homomorphism $(x,y)\rightarrow(x'',y'')$ as an isomorphic image of a subgroup of A under this uniformization. On the other hand, let e be the inseparability degree of the homomorphism $(x,y)\rightarrow(x'',y'')$. Then the e-th power of the factor group of A by that subgroup, which is obtained by raising the co-ordinates of points to their e-th powers [1], is an elliptic curve. We denote this elliptic curve by A' and the corresponding homomorphism by α. We are thus in the same situation as before; hence we get a homomorphism $(x,y)\rightarrow(x',y')$ of degree m of the Jacobi quartic of modulus ρ to the Jacobi quartic of modulus, say ρ', associated with A' with reference to $\pm\alpha r$. However, since the two homomorphisms $(x,y)\rightarrow(x',y')$ and $(x,y)\rightarrow(x'',y'')$ have the same kernel and the same inseparability degree, the image quartics must be isomorphic. We note that *the homomorphism* $(x,y)\rightarrow(x',y')$ *is uniquely determined by the given homomorphism* $(x,y)\rightarrow(x'',y'')$. In the following, we shall use the word "transformation" for this well-selected homomorphism of Jacobi quartics with variable moduli, i.e. with transcendental moduli over Q. Also,

we indicate the transformation by $(\rho, x, y) \to (\rho', x', y')$. If we polarize A and A' by $\pm r$ and $\pm r'$, the transformation is characterized up to the sign of x' by the commutativity of the following diagram

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d] & A' \arrow[d] \\ (\rho, x, y) \arrow[r] & (\rho', x', y'). \end{tikzcd}$$

This observation facilitates to verify that *a product of transformations is a transformation and that the multiplicator of the product is the product of multiplicators.* Moreover, if a transformation $(\rho, x, y) \to (\rho', x', y')$ is induced by α, then α' induces a transformation of the form $(\rho', x', y') \to (\rho, \pm x, y)$. The sign in front of x will be determined to be $(-4/m) = (-1)^{\frac{1}{2}(m-1)}$ in the next section. At any rate, we call it the *complementary transformation* of the given transformation, a terminology due to Jacobi. If μ is the multiplicator of the original transformation and μ' the one for the complementary transformation, we therefore get $\mu\mu' = (-4/m)m$. Now we shall prove the following theorem.

THEOREM 1. *Let $F(X)$ and $G(X)$ be the transformation polynomials associated with a transformation $(\rho, x, y) \to (\rho', x', y')$. Then the coefficients of $F(X)$ and $G(X)$ are contained in $Q(\rho, \rho')$.*

We first note that the assertion for $G(X)$ follows from the assertion for $F(X)$. In fact, if the coefficients of $F(X)$ are contained in $Q(\rho, \rho')$, the coefficients of $G(X)^2$, hence those of $G(X)$, must also be contained in the same field. We know by Chow's theorem that the coefficients of $F(X)$ are separable over $Q(\rho, \rho')$. Therefore, we have only to show that the kernel of the transformation is normally algebraic over $Q(\rho, \rho')$. If the kernel is not normally algebraic over $Q(\rho, \rho')$, as we can see, such a bad situation happens already by a transformation with cyclic kernel of the transformation degree not divisible by the characteristic. Therefore, we can assume that we have such a case from the beginning. However, this is not possible, because factor groups by distinct cyclic subgroups of the same order have even distinct absolute invariants [4]. This completes the proof.

Actually, in proving Theorem 1 we could have used the Galois theory of division points [4]. The proof would, then, become less formal. However, the above proof is of an elementary nature.

4. Multiplication of Jacobi quartics. If n is an odd natural number, the endomorphism n of A is of degree $m = n^2$ and maps r to $r' = (-4/n)r$. Therefore x' differs from x at most by sign and we have $\rho' = \rho$. In this case, the transformation is called a *multiplication* and we shall denote the corresponding polynomials by $F_n(X)$ and $G_n(X)$. We already encountered with a multiplication implicitly in the proof of Theorem 1. At any rate, $F_n(X)$ and $G_n(X)$ are polynomials of X with coefficients in $Q(\rho)$. Moreover, these polynomials are *universal* in the sense they are compatible with specializations. Here, as we agreed in Section 1, the reduction modulo 2 should be excluded. The following theorem is fairly deep and the crucial point of the proof is along Kronecker's idea.

THEOREM 2. *The coefficients of multiplication polynomials $F_n(X)$ and $G_n(X)$ are contained in $Z[8\rho]$.*

We can assume, because of the universality and of the nature of the theorem, that the characteristic of Q is zero. We shall first show that the assertion for $G_n(X)$ follows from the assertion for $F_n(X)$. In fact, the relation $dx(nu)/y(nu) = ndx/y$ shows that the coefficients of $nG_n(X)$ are contained in $Z[8\rho]$. On the other hand, $G_n(X)^2$ is a polynomial of X with coefficients in $Z[2\rho]$, hence in $Z[\rho]$. Since $G_n(X)$ satisfies $G_n(0) = 1$, by the so-called Gauss lemma, the coefficients of $G_n(X)$ itself must be contained in $Z[\rho]$. Therefore, the coefficients of $G_n(X)$ are contained in $Z[8\rho]$. In order to prove the assertion for $F_n(X)$, we must obtain the addition theorem for x. Incidentally, the subsequent discussion holds without the restriction on the characteristic. Let v and w be independent generic points of A over a common field of definition K of x and y. Put $v + w = u$ and take the field $K(x(u), y(u))$ as our domain of rationality. Then $x_1 = x(v)$ and $x_2 = x(w)$ are related by an irreducible symmetric equation which is quadratic in x_1 and in x_2 individually over this domain of rationality. If we make the translations $v \to v + 2s$ and $w \to w + 2s$ of A simultaneously, then x_1 and x_2 change their signs, hence the equation contains neither $x_1x_2(x_1 + x_2)$ nor $x_1 + x_2$. Furthermore, since we can make the translations $v \to v + r$ and $w \to w + r$ of A over the domain of rationality, the coefficient of $x_1^2x_2^2$ is equal to the constant term. Thus the equation takes the following form

$$\gamma_0(x_1^2x_2^2 + 1) + 2\gamma_1x_1x_2 + \gamma_2(x_1^2 + x_2^2) = 0.$$

Here, if we specialize v to u and w to o over the domain of rationality, we get $\gamma_0 + \gamma_2x^2(u) = 0$. Moreover, in order to solve the above equation in x_2, we must extract a quadratic radical from $\gamma_1^2x_1^2 - (\gamma_0 + \gamma_2x_1^2)(\gamma_0x_1^2 + \gamma_2)$.

Since x_2 must be contained in $K(x_1, y_1)$, therefore, this polynomial of x_1 must be a multiple of $1-2\rho x_1^2+x_1^4$. In this way, we see that the expression for x_2 takes the following form

$$x(u-v)=(x(u)y(v)-x(v)y(u))(1-x^2(u)x^2(v))^{-1}.$$

Once we have this formula, by specializing v to $-u$ over K we get $x(2u) =2xy(1-x^4)^{-1}$ with $x=x(u)$ and $y=y(u)$. Now, assume that the theorem is true up to n and specialize u to nu and v to $-2u$ over K. Then, if we observe $dx(2u)/y(2u)=2dx/y$ and $y(nu)=G_n(x)F_n(x)^{-2}y$, we see that $x((n+2)u)$ is of the form $N(x)D(x)^{-1}$ with

$$D(X)=(1-X^4)^2F_n(X)^2-4X^2(1-2\rho X^2+X^4)X^mF_n(X^{-1}).$$

The numerator $N(X)$ is, at any rate, a polynomial of X with coefficients in $Z[\rho]$. We observe that the coefficients of $D(X)$ are contained in $Z[8\rho]$ and $D(X)$ satisfies $D(0)=1$. Since $F_{n+2}(X)$ also satisfies $F_{n+2}(0)=1$ and divides $D(X)$ in the polynomial ring $Q(\rho)[X]$, by the Gauss lemma the division must take place within the coefficient domain $Z[8\rho]$. This completes the proof.

COROLLARY 1. *The coefficients of transformation polynomials $F(X)$ and $G(X)$ are integral over $Z[8\rho]$. Moreover, the roots of $G(X)=0$ are units over $Z[8\rho]$.*

The main assertion follows from the fact that $x(a)$ and $x(s+a)$ are integral over $Z[8\rho]$ by Theorem 2. The additional statement is a consequence of the fact that the highest coefficient of $G(X)$ as well as the constant term are 1. In connection with this, we shall show that the highest coefficient c_0 of the multiplication polynomial $F_n(X)$ is $(-4/n)n$. At any rate c_0 is the product of $x^2(a)$ extended over a half system. We are assuming, because of the universality, that the characteristic is zero. If we specialize ρ to 1, by a specialization argument, we see that n^2-n of the $x^2(a)$ specialize to 1 and c_0 will become a product of $x'^2(a')$, say, extended over a half system. Since the specialized addition theorem is of the form

$$x'(u'+v')=(x'(u')+x'(v'))(1+x'(u')x'(v'))^{-1},$$

we see that the product in question is $(-4/n)n$, as asserted. We also add the following corollary.

COROLLARY 2. *Let $(\rho, x, y)\to(\rho', x', y')$ be a transformation and put $\sigma=x(s)$ and $\sigma'=x'(\alpha s)$. Then $\sigma'\sigma^{-1}$ is a unit over $Z[8\rho]$, hence $2\rho'$ is integral over $Z[2\rho]$.*

In general, as we see from the addition theorem for x, the product $x(u+v)x(u-v)$ can be written in the form

$$(x^2(u)-x^2(v))(1-x^2(u)x^2(v))^{-1}.$$

Therefore, we can write σ' as follows

$$\begin{aligned}\sigma' &= (\prod_a (x(s)-x(a))(1-x(a)x(s))^{-1})^e \\ &= x^e(s)(\prod_{a \text{ half system}} (x^2(s)-x^2(a))(1-x^2(a)x^2(s))^{-1})^e \\ &= \sigma^e \prod_{a \text{ half system}} x^{2e}(s+a).\end{aligned}$$

On the other hand, σ is a unit over $Z[2\rho]$ as a root of $1-2\rho X^2+X^4=0$ and, of course, we have $e=1$ in the case of characteristic zero. The corollary follows immediately from these facts and from the above expression for σ'.

5. A digression. In reviewing known results in Section 1, we did not mention properties of *Hasse invariant*. Actually, Kronecker seemed to know the full meaning of this invariant, which we shall discuss by a slightly different purely algebraic method. In general we can expand $(1-2\rho x^2+x^4)^{-\frac{1}{2}}$ in a formal power-series of x as follows

$$(1-2\rho x^2+x^4)^{-\frac{1}{2}}=\sum_{i=0}^{\infty} P_i(\rho)x^{2i}.$$

The coefficients $P_0(\rho)=1, P_1(\rho),\cdots$ are the so-called *Legendre polynomials.* If the characteristic is zero, we can integrate the differential equation $dt=dx/y$ by a formal power-series $t=x+\cdots$. This power-series can be inverted and we get $x=t+\cdots$. Call it $f(t)$. Then we can expand $f(nt)$ in a formal power-series of x. Also, we can expand $x(nu)$ in a formal power-series of $x=x(u)$. The addition theorem of invariant differentials guarantees that these two expansions are the same. If n is a prime number p, from this we can derive congruence properties of the coefficients of the multiplication polynomial $F_p(X)$. For instance, it is quite easy to get the following congruence relations

$$c_i\equiv(-4/n)(p/(2i+1))P_i(\rho) \bmod p^2$$

for $i=0,1,\cdots,\frac{1}{2}(p-1)$. In particular, if we take the last congruence relation modulo p, we get

$$\prod_{\substack{pa=0\\ a\neq 0}} x^p(a)=(-4/n)P(\rho)$$

with $P(\rho)=P_{\frac{1}{2}(p-1)}(\rho)$. Therefore, we have $P(\rho)=0$ if and only if A has

no point of order p except o. On the other hand, since $P(\rho)$ is the $\frac{1}{2}(p-1)$-th Legendre polynomial, it satisfies the Legendre differential equation, which in the case of characteristic p, takes the following form

$$(1-\rho^2)d^2P/d\rho^2-2\rho dP/d\rho-(\tfrac{1}{4})P=0.$$

Here $P(\pm 1)$ is different from zero; hence *from the differential equation it follows that the* $\frac{1}{2}(p-1)$ *roots of* $P(\rho)=0$ *are simple.* The Galois group of the ρ-space over the j-space operates on this set of $\frac{1}{2}(p-1)$ roots and we can count the number of orbits quite easily using results in Section 2. This number is precisely the number of non-isomorphic elliptic curves in characteristic p having no point of order p except neutral element. We have already discussed this topic, including the meaning of the coefficient $\frac{1}{4}$, in a separate paper [5]. At any rate, this was how we got the well-known Gauss differential equation for an explicit form of Hasse invariant calculated by Deuring [2] using Hasse's theory.

If we also take into account the fact that $P(\rho)=0$ implies $F_p(X)=1$, we see that the non-vanishing coefficients of $F_p(X)$ in characteristic p are divisible by $P(\rho)$. In other words, we can write $F_p(X)$ in the form

$$F_p(X)=1+\sum_{0<2i<p-1}P(\rho)\gamma_i(\rho)X^{(p-2i-1)p}+(-4/p)P(\rho)X^{(p-1)p}$$

with certain polynomials $\gamma_1(\rho),\gamma_2(\rho),\cdots$ of ρ with coefficients in Z. There is a formal analogy between this relation and the Kronecker conguence relation which we shall discuss in the last section.

6. Absolute irreducibility of certain equations. We come to the section where our geometric theory of modular functions plays a role. As before, let n be an odd natural number and assume that the modulus ρ is transcendental over Q. We assume, in addition, that n is not divisible by the characteristic. If a is a point of A of order n, then $x(a)$ is integral over $Z[8\rho]$. Moreover, if we put

$$\Phi_n(X)=\prod_{a\text{ primitive}}(X-x(a)),$$

then $\Phi_n(X)$ is a polynomial of X^2 with coefficients in $Q(\rho)$. The coefficients are, therefore, contained in $Z[8\rho]$, hence we can write $\Phi_n(X)$ in the form $\Phi_n(X^2,8\rho)$ with a polynomial $\Phi_n(X,Y)$ of X and Y with coefficients in Z. We call $\Phi_n(X,Y)=0$ the n-th *division equation.* In the simplest case $n=3$, the division equation reads as $X^4-6X^2+XY-3=0$. In general, if we denote the Euler function by $\phi(n)$ and the number of cyclic subgroups of

order n in the group of points of A of order n by $\psi(n)$, we can easily calculate the degree and the constant term of $\Phi_n(X)$ and we get

$$\deg \Phi_n(X) = \phi(n)\psi(n) = n^2 \prod_{p \mid n} (1 - p^{-2})$$

$$\Phi_n(0) = \begin{cases} (-4/n)p & n = p^e \\ (-4/n) & \text{otherwise.} \end{cases}$$

Here p means a prime number. We can use, for instance, the so-called Möbius inversion formula to make the calculation. At any rate, *in case n contains two distinct prime factors, the roots of $\Phi_n(X) = 0$ are units over $Z[8\rho]$*. Now, we shall prove the following theorem.

THEOREM 3. *The division equation $\Phi_n(X, Y) = 0$ is absolutely irreducible.*

Let λ' be transcendental over Q and consider the plane cubic $Y^2 = X(1-X)(\lambda'-X)$ with $(0,1,0)$ as neutral element. We take this elliptic curve as A. Let Ω be the group of points of A of order n. Also, let k be the algebraic closure of Q and adjoin the points of A of order four to $k(\lambda')$. Then we get an extension K of $k(\lambda')$. The Galois group of $K(\Omega)$ over K operates on Ω and, with respect to a base of Ω, we get a representation by two-by-two integer matrices of determinant 1 modulo n. The point is that we get all such matrices in this way [4, Theorem 1]. On the other hand, x^2 is defined over K and ρ is contained in K. Moreover ρ is certainly transcendental over Q. Since the only conjugates of a over $K(x^2(a))$ are a and $-a$, we see that the Galois group of $K(x^2(a))_{na=0}$ over K is what we denoted by $LF(2, n)$ in [4], which is the whole group of two-by-two integer matrices of determinant 1 modulo n divided by plus and minus of the unit matrix. As a consequence, the Galois group operates transitively on the roots of $\Phi_n(X, 8\rho) = 0$. Therefore $\Phi_n(X, 8\rho)$ is irreducible over K. This is already much stronger than Theorem 3.

There are two significant equations which are "derived" successively from the division equation. Consider a transformation $(\rho, x, y) \to (\rho', x', y')$ of degree n with cyclic kernel. Since we have $\psi(n)$ possibilities for the kernel, we have the same number of distinct moduli ρ'. Since $2\rho'$ is integral over $Z[2\rho]$, we get a polynomial $\Psi_n(X, Y)$ of X and Y with coefficients in Z of degree $\psi(n)$ in X satisfying

$$\Psi_n(X, 2\rho) = \prod_{\rho'} (X - 2\rho').$$

We call $\Psi_n(X, Y) = 0$ the n-th *modular equation*. The existence of the

complementary transformation and the irreducibility of $\Psi_n(X, Y)$ which we shall prove presently show that $\Psi_n(X, Y)$ is symmetric in X and Y. For example, the third modular equation reads as follows

$$(X^3 - Y)(X - Y^3) - 3[5X^3Y^3 + 2.31X^2Y^2 + 5.17XY$$
$$- 2^2 5XY(X^2 + Y^2) - 2^8(X^2 + Y^2) + 2^{10}] = 0.$$

This suggests that modular equations have less complicated numerical coefficients than "invariant transformation equations."

On the other hand, if μ is the multiplicator of the transformation $(\rho, x, y) \rightarrow (\rho', x', y')$, since μ is integral over $Z[8\rho]$, we get a polynomial $M_n(X, Y)$ of X and Y with coefficients in Z of degree $\psi(n)$ in X satisfying

$$M_n(X, 8\rho) = \prod_{\rho'} (X - \mu).$$

We do not know apriori that the $\psi(n)$ roots of $M_n(X, 8\rho) = 0$ are distinct. At any rate, we call $M_n(X, Y) = 0$ the n-th *multiplicator equation.* The third multiplicator equation reads as $X^4 - 6X^2 - XY - 3 = 0$. In general, the constant term $M_n(0, 8\rho)$, which will turn out to be the norm of μ taken from $Q(\rho, \mu)$ to $Q(\rho)$, can be calculated without difficulty and we get

$$M_n(0, Y) = \prod_{p \mid n} (-4/p)^{\frac{1}{2}e(e+1)} p^{E_p}$$

with

$$E_p = ((p^e - 1)/(p - 1))\psi(n/p^e).$$

Here p^e is the contribution of p to n. In particular, for $n = p$ the constant term is simply $(-4/p)p$. We note that $\Phi_n(X, Y)$, $\Psi_n(X, Y)$ and $M_n(X, Y)$ are universal like $F_n(X)$ and $G_n(X)$. We shall now prove the following theorem.

THEOREM 4. *The modular equation $\Psi_n(X, Y) = 0$ and the multiplicator equation $M_n(X, Y) = 0$ are both absolutely irreducible.*

We know that we have $\psi(n)$ distinct transformed moduli ρ'. Moreover, if we use the same notation as in the proof of Theorem 3, we see that the Galois group of $K(x^2(a))_{na=0}$ over K operates transitively on these moduli. This proves the absolute irreducibility of the modular equation. The same proof can be applied to the multiplicator equation once we are sure that the $\psi(n)$ multiplicators are distinct. Suppose that two distinct transformations $(\rho, x, y) \rightarrow (\rho', x', y')$ and $(\rho, x, y) \rightarrow (\rho'', x'', y'')$ have the same multiplicator. Then the complementary transformation of the first multiplied by the second is a transformation of the form $(\rho', x', y') \rightarrow (\rho'', \pm x'', y'')$ with $\pm n$ as multi-

plicator. If the kernel of this transformation is not cyclic, a multiplication can be factored. In this way, by changing the notation, we are led to the situation where the $\psi(n)$ transformations $(\rho, x, y) \rightarrow (\rho', x', y')$ have the same rational multiplicator m, i.e.

$$m = \prod_{i=1,\cdots,n-1} x(ia) = (-4/n) \prod_{i=1,3,\cdots,n-2} x^2(ia)$$

for some element m of Z. However, there certainly exists an extension of the specialization $\rho \rightarrow \infty$ over Q in which $x(a)$; hence $x(ia)$ for $i = 1, 3, \cdots, n-2$, are specialized to ∞. This is a contradiction. Therefore, the multiplicators are distinct and the multiplicator equation is absolutely irreducible.

COROLLARY. *If μ is the multiplicator of the transformation $(\rho, x, y) \rightarrow (\rho', x', y')$, we have $Q(\rho, \rho') = Q(\rho, \mu)$.*

7. Kronecker's congruence relation. Assume now that the degree of the transformation $(\rho, x, y) \rightarrow (\rho', x', y')$ is an odd prime number p different from the characteristic of Q. Then we can prove the following theorem.

THEOREM 5. *The transformation polynomial $F(X)$ for the transformation of degree p can be written in the form*

$$F(X) = 1 + \sum_{0<2i<p-1} \mu\gamma_i X^{p-2i-1} + \mu X^{p-1}.$$

Here $\gamma_1, \gamma_2, \cdots$ are integers of $Q(\rho, \mu)$ with reference to $Z[8\rho]$. Moreover, the norm of μ taken from $Q(\rho, \mu)$ to $Q(\rho)$ is $(-4/p)p$.

We proved the additional part in a more general form in the previous section. As for the main assertion, it is trivial in the case of positive characteristic, because μ is then a unit of $Q(\rho, \mu)$ with reference to $Z[8\rho]$. Therefore, we shall assume that the characteristic is zero. Then, the norm of μ taken from $Q(\rho, \mu)$ to $Q(\rho)$ generates a prime ideal in $Z[8\rho]$. Therefore, from the ideal theory of integrally closed Noetherian domains [10], we can conclude that μ generates a prime ideal in the principal order of $Q(\rho, \mu)$ with reference to $Z[8\rho]$. Moreover, we know that the coefficients $c_1, c_2, \cdots$ of $F(X)$ are elements of the same principal order. Since there exists a reduction modulo p in which $x(ia)$ for $i = 1, \cdots, p-1$, and μ are mapped to zero, while ρ remains to be a variable modulus, we see that $c_1, c_2, \cdots$ are multiples of μ. In other words, if we write $c_1, c_2, \cdots$ in the form $\mu\gamma_1, \mu\gamma_2, \cdots$, then $\gamma_1, \gamma_2, \cdots$ must belong to the principal order of which we are talking. This completes the proof.

In a slightly ambiguous way, the main part of Theorem 5 can be expressed

simply by the following congruence relation

$$x'(\alpha u) \equiv x^p(u) \quad \mathrm{mod}\, \mu.$$

This is the *Kronecker congruence relation.* As he himself confirmed, this congruence relation has a fundamental meaning for the entire transformation theory of elliptic functions as well as its arithmetic applications.

We add a remark about the connection between Theorem 5 and a formula we obtained in Section 5. Let μ' be the multiplicator of the compelementary transformation $(\rho', x', y') \to (\rho, (-4/p)x, y)$ of $(\rho, x, y) \to (\rho', x', y')$. Then, as we know more generally, we have $\mu\mu' = (-4/p)p$. In particular, μ and μ' are both elements of $Q(\rho, \rho')$. Now, we know that the relation between ρ and ρ' is symmetric. Since $2\rho'$ is integral over $Z[2\rho]$, and conversely, we see that the principal order of $Q(\rho, \rho')$ with reference to $Z[2\rho]$ is also the principal order of $Q(\rho, \rho')$ with reference to $Z[2\rho']$. Apparently, this principal order contains the principal orders of $Q(\rho, \rho')$ with reference to $Z[8\rho]$ and $Z[8\rho']$. Therefore, by the same reason as in the proof of Theorem 5, we see that μ and μ' generate prime ideals in that principal order. These prime ideals are distinct, because they even have different norms taken from $Q(\rho, \rho')$ to $Q(\rho)$, say. Now, if $F_p(X)$ is the multiplication polynomial and if we reduce $F_p(X)$ modulo μ', we get

$$F_p(X) \equiv 1 + \sum_{0<2i<p-1} \mu^p \gamma_i{}^p X^{(p-2i-1)p} + \mu^p X^{(p-1)p} \quad \mathrm{mod}\, \mu'.$$

If we compare this relation with the formula in Section 5, we get, for instance, the following relation

$$\mu^p \equiv (-4/p) P(\rho) \quad \mathrm{mod}\, \mu'.$$

Finally, we add some remarks concerning the first non-trivial case $p = 5$ for Theorem 5. We know that Kronecker already mentioned this case in [8]. However, our precise theory contradicts some of his calculations. Thus, in order to avoid misunderstandings, we must point out his mistakes. First, the fifth division equation should read as follows

$$\begin{aligned} &X^{12} - 50X^{10} + 35YX^9 - (10Y^2 + 125)X^8 + (Y^3 + 92Y)X^7 \\ &\quad - (15Y^2 + 300)X^6 + 90YX^5 - 105X^4 - 20YX^3 \\ &\quad + (Y^2 + 62)X^2 - 5YX + 5 = 0. \end{aligned}$$

Thus, by reduction modulo 5 we get $X^2(X^{10} + (Y^2 + 2)YX^5 + (Y^2 + 2)) = 0$. In Kronecker [8, p. 454], the coefficient of X^7 reads, in our notation, as

$Y^3 + 368$, which seems to be a misprint. Second, the equation for $\gamma = \gamma_1$ should read as follows

$$\gamma^6 + 8\rho \cdot \gamma^5 + 20\gamma^4 - 80\gamma^2 - 16 \cdot 8\rho \cdot \gamma - (8\rho)^2 = 0.$$

Thus γ is, indeed, integral over $Z[8\rho]$. In Kronecker [8, p. 452], intermediate terms seem to be miscalculated.

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FIBRE SYSTEMS OF JACOBIAN VARIETIES.*[1]

(III. Fibre Systems of Elliptic Curves.)

By Jun-ichi Igusa.

To Zariski on his 60th birthday

1. Introduction. This is the third paper on fibre systems of Jacobian varieties. We shall apply some of our previous results [7, 9] to the simplest case of fibre systems of elliptic curves. In this way we get a *geometric theory of elliptic modular functions* with arbitrary level in the sense of Klein [11] for any characteristic which does not divide the level. Here we would like to state that no algebraic theory of modular functions has been available even in the case of characteristic zero. In fact, the known theory depends heavily either on Riemann's existence theorem or on the use of Eisenstein series.

The theory of modular functions in positive characteristic is useful in discussing arithmetic properties of modular functions in characteristic zero. For instance, we can solve the so-called "generalized Ramanujan conjecture" in a *definitive form* for dimension -2. We are following here the general method, orginated by Galois, for constructing a theory in positive characteristic and applying this theory to problems in characteristic zero. In addition to the above mentioned application, which we shall discuss in a separate paper, our theory also contributes to the "three points ramification problem." In fact, we can now construct canonically a Galois covering of a straight line which is ramified at either three or in some cases two points and having $LF(2, n)$ as Galois group, provided n is not a multiple of the characteristic. The reader will find more precise statements in Theorem 4 and Theorem 6. We note that, at least in the case of characteristic zero, these theorems characterize the field of modular functions uniquely. We also note that Gierster's theorem which shows that $LF(2, p)$ contains the tetrahedral group for any odd prime number p is automatically contained in Theorem 6.

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2. Absolute invariant. Let A be an elliptic curve, i.e. an Abelian variety of dimension 1. We are going to associate a finite quantity $f(A)$ to A, which is contained in the universal domain of A, so that f satisfies the following conditions:

(1) *We have* $f(A)=f(B)$ *if and only if* A *and* B *are isomorphic;*

(2) *If* A' *is a specialization of* A, *then* $f(A')$ *is the unique specialization of* $f(A)$ *over this specialization.*

Here we are assuming, of course, that B and A' are elliptic curves. Moreover, unequal-characteristic specialization [cf. 14] is also included in the second condition. In the following we shall construct f by examining *normal forms* of elliptic curves due to *Legendre* and *Hesse*. As the reader will see, the construction is not quite arbitrary and it turns out that f is unique up to the transformation $f\rightarrow \pm f+\text{integer}$.[2] Observe beforehand that $f(A)$ must be contained in every field of definition of the "underlying" curve of A.

To begin with, we shall recall a *theorem of Hasse*: Let n be a natural number and let u be a typical point of an elliptic curve A. Then the mapping $u\rightarrow n\cdot u$, which is an endomorphism of A, is of degree n^2. The mapping in question is separable if and only if n is not a multiple of the characteristic. In this case, therefore, the kernel is a finite Abelian group of type (n,n). However, in case n is a multiple of the characteristic, the kernel has a different structure depending on the behavior of the Kronecker-Hasse invariant of A [5, 13]. At any rate, let $\mathfrak{o}$ be the neutral element of A. Then the complete linear system on A determined by $3\cdot\mathfrak{o}$ gives rise to an isomorphism of A to a plane cubic such that the image of $\mathfrak{o}$ is a flection point. Therefore we can assume from the beginning that A is a plane cubic with a flection point as its neutral element. Then three points u, v, w of A satisfy $u+v+w=\mathfrak{o}$ if and only if they are collinear. Therefore, as we can see, if B is isomorphic to A, the isomorphism is induced by a linear collineation.

In case the characteristic is different from 2 and only in such case A contains three points of order 2 besides $\mathfrak{o}$ by the above mentioned theorem of Hasse. Moreover these three points are collinear. Therefore we can take the straight line passing through these points as X-axis. Also we take the flection tangent at $\mathfrak{o}$ as Z-axis, i.e. the line at infinity, and a line joining $\mathfrak{o}$ to one of the three points of order 2 as Y-axis. This is possible, because these three straight lines can not go through one same point. Furthermore we can assume

[2] The possibility of unique characterization was suggested by Professor Weil when the author met him at Princeton in January, 1958.

that one of the two remaining points of order 2 has co-ordinates $(1,0,1)$. Here we agree once for all to write co-ordinates in the order X, Y, Z. If the third point has co-ordinates $(\lambda,0,1)$, the equation of the cubic reads as $\alpha\cdot Y^2Z = X(X-Z)(X-\lambda Z)$ with $\alpha \neq 0$. If we replace $\alpha^{\frac{1}{2}}\cdot Y$ by Y, we can write the equation as follows

$$Y^2Z = X(X-Z)(X-\lambda Z).$$

Here the *modulus* λ is different from 0, 1, ∞ and conversely, if that is so, the above equation defines a non-singular cubic. This geometric consideration shows that we have twelve choices for the normalized co-ordinate systems and, accordingly, we get the following six values for the moduli

$$\begin{array}{ccc} \lambda & 1-\lambda & (\lambda-1)\lambda^{-1} \\ \lambda^{-1} & (1-\lambda)^{-1} & \lambda(\lambda-1)^{-1}. \end{array}$$

More precisely, the corresponding six cubics, with $(0,1,0)$ as o, are isomorphic and the twelve isomorphisms can be given inhomogeneously by the corresponding generic points as follows

$$\begin{array}{ll} (x, \pm y) & (1-x, \pm(-1)^{\frac{1}{2}}\cdot y) \\ (x\lambda^{-1}, \pm \lambda^{-\frac{3}{2}}\cdot y) & ((1-x)(1-\lambda)^{-1}, \pm(\lambda-1)^{-\frac{3}{2}}\cdot y) \end{array}$$
$$((\lambda-x)\lambda^{-1}, \pm(-\lambda)^{-\frac{3}{2}}\cdot y)$$
$$((\lambda-x)(\lambda-1)^{-1}, \pm(1-\lambda)^{-\frac{3}{2}}\cdot y).$$

We can consider those six values of moduli as defining a transformation group operating on a projective straight line. The orbits consist of six points in general except for the *degenerate case* $\{0,1,\infty\}$, the *harmonic case* $\{-1,\frac{1}{2},2\}$ and the *equianharmonic case* $\{-\rho,-\rho'\}$ with respective multiplicities 2, 2 and 3. Here ρ and ρ' are primitive third roots of unity. We note that our group is operating on $\{0,1,\infty\}$ as symmetric group. According to Lüroth's theorem, we can identify these orbits to single points of a projective straight line. Since we have three freedoms at our disposal, we can do this in such a way that the first and the third exceptional orbits are mapped respectively to ∞ and 0. Then up to a constant factor the identification mapping has the following form

$$j = 2^8(\lambda^2-\lambda+1)^3 : \lambda^2(\lambda-1)^2.$$

This j is called the *absolute invariant* of A. We note that the second exceptional orbit is mapped to 12^3. In case the characteristic is 3 the situation is slightly different. We have only two exceptional orbits $\{0,1,\infty\}$ and $\{-1\}$

with respective multiplicities 2 and 6. However, apparently the same j as above works also in this case. We also note that the Lüroth theorem was used only to motivate the reasoning.

In the case of characteristic 2 or, more generally, if the characteristic is different from 3 and only in such case A contains nine flection points, i.e. points of order 3 by Hasse's theorem. As before, let ρ and ρ' be the primitive third roots of unity. Then, as we can see, it is possible to choose a co-ordinate system so that the nine flection points have the co-ordinates $(0,-1,1)$, $(0,-\rho,1)$, $(0,-\rho',1)$; $(1,0,-1)$, $(\rho,0,-\rho')$, $(\rho',0,-\rho)$; $(-1,1,0)$, $(-1,\rho',0)$, $(-1,\rho,0)$. Consequently the equation of the cubic reads as

$$X^3+Y^3+Z^3=3\mu XYZ.$$

Here the *modulus* μ is different from 1, ρ, ρ', ∞ and conversely, if that is so, the above equation defines a non-singular cubic. If we agree to take $(-1,1,0)$ as 0, we have four choices for Z-axis. Then the other axes are determined up to a permutation, which reflects the automorphism $u\to -u$ of A. Since Z can be multiplied by an arbitrary third root of unity, we have twenty-four choices for the normalized co-ordinate systems and, accordingly, we get the following twelve values for the moduli

$$\zeta\mu \qquad \zeta(\mu+2\zeta')(\mu-\zeta')^{-1}.$$

The twenty-four isomorphisms themselves can be given inhomogeneously, up to permutations of co-ordinates, by the corresponding generic points as follows

$$(\zeta x, \zeta y)$$
$$(\zeta(\rho x+\rho' y+\zeta')(x+y+\zeta')^{-1}, \zeta(\rho' x+\rho y+\zeta')(x+y+\zeta')^{-1}).$$

Here ζ and ζ' are the third roots of unity. We can consider those twelve values of moduli as defining a transformation group operating on a projective straight line. The orbits consist of twelve points in general except for the *degenerate case* $\{1,\rho,\rho',\infty\}$, the *harmonic case* $\{1\pm 3^{\frac{1}{2}}, (1\pm 3^{\frac{1}{2}})\rho, (1\pm 3^{\frac{1}{2}})\rho'\}$ and the *equianharmonic case* $\{0,-2,-2\rho,-2\rho'\}$ with respective multiplicities 3, 2 and 3. We note that our group is operating on $\{1,\rho,\rho',\infty\}$ as alternating group. Again we can identify these orbits to single points of a projective straight line so that the first and the third exceptional orbits are mapped respectively to ∞ and 0. Then up to a constant factor the identification mapping has the following form

$$j=3^3\mu^3(\mu^3+2^3)^3:(\mu^3-1)^3.$$

This j is called the *absolute invariant* of A. We note that the second excep-

tional orbit is mapped to 12^3. In case the characteristic is 2 the situation is slightly different. We have only two exceptional orbits $\{1, \rho, \rho', \infty\}$ and $\{0\}$ with respective multiplicities 3 and 12. However, apparently the same j as above works also in this case. We note that in case the characteristic is different from 2 and 3 both normalizations are possible. In this way we get an algebraic correspondence between λ-space and μ-space. If λ and μ correspond to each other, they give rise to one and the same j. The correspondence $A \rightarrow j(A)$, therefore, satisfies the condition (1) stated in the beginning. Moreover, it is a simple exercise to show that j also satisfies the condition (2). As for the general expression of a correspondence f in the form $f = \pm j +$ integer, it is obvious from the construction of j.

There is another normal form, due to Deuring [3], which is valid provided A contains at least one flection point besides o. This is certainly the case if the characteristic is different from 3, and we shall assume this to be satisfied. Take the line joining these two flection points as Y-axis and the flection tangents as two other axes so that o is represented by $(0,1,0)$. If we normalize the co-ordinate system so that the third flection point on the Y-axis has co-ordinates $(0,1,1)$, the equation of the cubic reads as $Y^2Z - YZ^2 = \beta \cdot X^3 + 3\nu XYZ$ with $\beta \neq 0$. If we replace $\beta^{\frac{1}{3}} \cdot X$ by X, we can write the equation as follows

$$Y^2Z - YZ^2 = X^3 + 3\nu XYZ.$$

Here the *modulus* ν is different from 1, ρ, ρ', ∞ and conversely, if that is so, the above equation defines a non-singular cubic. Furthermore, for instance, if we make the co-ordinate transformation

$$X : Y : Z \rightarrow \rho\mu\nu^{-1}Z : X + \rho'Y + \rho\mu Z : -\rho(X + Y + \mu Z),$$

then $Y^2Z - YZ^2 = X^3 + 3\nu XYZ$ is transformed into $X^3 + Y^3 + Z^3 = 3\mu XYZ$. Here μ and ν are related by $(\mu^3 - 1)(\nu^3 - 1) = 1$. Therefore the absolute invariant can be expressed in terms of ν as follows

$$j = 3^3\nu^3(3^3\nu^3 - 2^3)^3 : \nu^3 - 1.$$

We note that this relation considered as an equation for ν is not normal. This is one great disadvantage of the normal form $Y^2Z - YZ^2 = X^3 + 3\nu XYZ$. However this normal form degenerates in a much nicer way than the normal form $X^3 + Y^3 + Z^3 = 3\nu XYZ$ in the sense we shall explain later.

3. Fields of modular functions of low levels. Suppose that A is an elliptic curve whose absolute invariant $j(A)$ is a variable over a prime field F.

Consider a homomorphism of A onto another elliptic curve B. Since every homomorphism can be factored into homomorphisms of prime degrees, we can assume that the degree of the homomorphism is a prime number p. Also we can assume that p is different from the characteristic of F, for otherwise the situation will become trivial. Then the *transformation theory of elliptic functions* will answer to the following problems: (1) How is $j(B)$ related to $j(A)$? (2) What is the arithmetic nature of the homomorphism? First part is the transformation of the parameter space while the second part is the transformation of the entire fibre system of elliptic curves. At any rate the homomorphism is determined up to an isomorphism by its kernel, which is a cyclic subgroup of A of order p. We know that the points of A of order p form a finite Abelian group of type (p,p). Therefore we have $p+1$ distinct homomorphisms from A to another elliptic curves of which B is a member. This already suggests the importance of points of finite orders. In fact the Galois theory of these points is a necessary preliminary for the transformation theory. In this section we shall define the field of modular functions with level along this line and we shall investigate the special cases of level 2 and of level 3. We shall start by summarizing some of the well-known elementary properties of groups of two-by-two integer matrices modulo integers.

Let n be a non-negative integer different from 1. The case $n=0$ is exceptional. We shall denote the general linear group of two-by-two matrices over integers modulo n by $GL(2,n)$. Similarly the special linear group $SL(2,n)$ is defined. We shall use the following notation for the simplicity of the printing

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a\ b, c\ d).$$

The group $SL(2,n)$ contains the group Z consisting of $\pm(1\ 0, 0\ 1)$ in the center. The linear fractional group $LF(2,n)$ is defined as the corresponding factor group. We note that Z can reduce to the identity in case $n=2$ and only in this case. If n splits into a product of relatively prime natural numbers n' and n'', then $GL(2,n)$ and $SL(2,n)$ decomposes into direct products $GL(2,n')\times GL(2,n'')$ and $SL(2,n')\times SL(2,n'')$. Therefore, if $\phi(n)$ is the Euler function and if $\psi(n)$ denotes the number of cyclic subgroups of order n in the Abelian group of type (n,n), i.e., if we put

$$\psi(n) = n\cdot\prod_{p\mid n}(1+p^{-1}),$$

the orders of $GL(2,n)$, $SL(2,n)$ and $LF(2,n)$ can be expressed as $n\phi(n)^2\psi(n)$, $n\phi(n)\psi(n)$ and $\frac{1}{2}\cdot n\phi(n)\psi(n)$. Here the case $n=2$ is exceptional for $LF(2,n)$

and its order is equal to $n\phi(n)\psi(n) = 6$. Moreover, if we exclude this case, the number of elements of order 2 in $SL(2,n)$ is smaller than the number of elements of order 2 in $LF(2,n) \times Z$. Hence $SL(2,n)$ does not split into this direct product. We also note that $(1\ 1, 0\ 1)$ and $(1\ 0, 1\ 1)$ generate $SL(2,0)$, hence $SL(2,n)$ in general. In this connection we note that the normal subgroup of $SL(2,0)$ of index 6 defined by $(a\ b, c\ d) \equiv (1\ 0, 0\ 1)$ mod 2 is the direct product of Z and the subgroup of $SL(2,0)$ generated by $(1\ 2, 0\ 1)$ and $(1\ 0, 2\ 1)$, which is a free group. A similar situation prevails even if we take module 2^e with $e \geqq 2$. Also, if a subgroup of $SL(2,3^e)$ reduces to the whole $SL(2,3)$ modulo 3 and contains both $(1\ 3, 0\ 1)$ and $(1\ 0, 3\ 1)$, this group must be $SL(2,3^e)$ for $e \geqq 1$. A proof can be obtained by induction on e.

Let A be an elliptic curve defined over a field K. Let n be a natural number not divisible by the characteristic of K and let Ω be the group of points of A of order n. Then Ω is a finite Abelian group of type (n,n) and $K(\Omega)$ is a finite Galois extension of K. Moreover the Galois group of $K(\Omega)$ over K operates faithfully on Ω. Therefore the Galois group is mapped isomorphically into $GL(2,n)$ with reference to a base of Ω. As we know, if we denote by k_0 the field of n-th roots of unity defined over the prime field F, then $K(\Omega)$ always contains k_0. Moreover, if an automorphism of $K(\Omega)$ over K transforms n-th roots of unity to m-th powers, the determinant of its image in $GL(2,n)$ is congruent to m modulo n. In particular, if K contains k_0, the image of the Galois group in $GL(2,n)$ is contained in $SL(2,n)$. We note that these properties come from the existence of a skew-symmetric pairing of Ω to itself [16, 9]. We also note that the classical proof depends on the so-called Abel's relation [12, 15].

Now, suppose that the absolute invariant $j(A)$ of A is different from 0 and 12^3. Then A has only two automorphisms defined by $u \to \pm u$. This is clear from the consideration made in the previous section. Suppose next that A' is another elliptic curve, also defined over K, with $j(A') = j(A)$. Then, by the above remark, we have only two isomorphisms between A and A', which we can denote by α and $-\alpha$. In general α is not defined over K. However α is always defined over a separable extension of K. This is a consequence of a general theorem of Chow for Abelian varieties [2], and in the present case it can be proved elementarily using the results in the previous section. These being remarked, if we identify points u and $-u$ on A, we get a projective straight line D defined over K. We shall denote the identification mapping, also defined over K, by Ku. Suppose that D' is associated with A' as D is associated with A. Then we get a correspondence between D and D'

as follows: Let u be a generic point of A over K. Then $Ku(u)$ and $Ku(\pm\alpha u)$ are the corresponding generic points of D and D' over K. We note that $Ku(\pm\alpha u)$ has no other conjugate over $K(Ku(u))$ than itself. Moreover $Ku(\pm\alpha u)$ is separable over $K(Ku(u))$, hence $Ku(\pm\alpha u)$ is rational over $K(Ku(u))$. Since the situation is symmetric in A and A', we see that the correspondence between D and D' is an everywhere biregular mapping defined over K. As a consequence, if v and v' are the corresponding points of A and A' under one of the two isomorphisms between A and A', we always get $K(Ku(v))=K(Ku(v'))$. This fact will be of constant use later.

Now, let j be a variable over the prime field F. Then we can find an elliptic curve A_j defined over $F(j)$ with $j(A_j)=j$. The most elementary proof can be obtained by just writing down equations for elliptic curves having j as absolute invariant. In case the characteristic is different from 2 and 3 we take

$$Y^2Z=4X^3-3^3j(j-12^3)^{-1}(X+Z)Z^2$$

as A_j. In case the characteristic is 2 or 3 we take

$$Y^2Z-XYZ=j^{-1}X^3+jZ^3, \qquad Y^2Z=X^3-X^2Z+j^{-1}Z^3$$

accordingly as A_j [3]. These cubic curves are indeed non-singular, hence we can introduce group structures with reference to $(0,1,0)$. We note that these elliptic curves remain to be elliptic curves with j as absolute invariant provided j is different from 0, 12^3, ∞. We shall see later that elliptic curves like A_j must degenerate at $j=0$, 12^3 and ∞ irrespective of its construction.

As before, let n be a natural number not divisible by the characteristic of F and let Ω be the group of points of A_j of order n. Then $F(j,\Omega)$ is a finite Galois extension of $F(j)$ and so is $F(j,Ku(\Omega))$. *This Galois extension of $F(j)$ is intrinsically defined by n,* i.e. it does not depend on the choice of A_j. We extend F to its algebraic closure k and we call $k(j,Ku(\Omega))$ the *field of modular functions of level n.* We note that $k(j,\Omega)$ is at most a quadratic extension of $k(j,Ku(\Omega))$. Also, if m is a natural number, the endomorphism $u\rightarrow m\cdot u$ of A_j gives rise to a natural projection of the field of modular functions of level mn to the field of modular functions of level n. We shall show that $k(\lambda)$ *and* $k(\mu)$ *are the fields of modular functions of level* 2 *and of level* 3. Here λ and μ correspond to j by the equations discussed in the previous section.

Let Ω be the group of points of A_j of order 2. We assume, of course, that the characteristic is different from 2. Then we have $F(j,Ku(\Omega))=F(j,\Omega)$ and we can transform A_j to $\alpha\cdot Y^2Z=X(X-Z)(X-\lambda Z)$ over $F(j,\Omega)$.

Therefore $F(\lambda)$ is contained in $F(j,\Omega)$. However, since $[F(j,\Omega):F(j)]$ is at most equal to 6, we have $F(j,\Omega)=F(\lambda)$, hence $k(j,Ku(\Omega))=k(\lambda)$. Let Ω be, next, the group of points of A_j of order 3. We are automatically assuming that the characteristic is different from 3. Then we can transform A_j to $X^3+Y^3+Z^3=3\mu XYZ$ over $F(j,\Omega)$, hence $F(\mu,\rho)$ is contained in $F(j,\Omega)$. Here ρ is, as before, a primitive third root of unity. On the other hand, because of a property of Ku, we get $F(\mu,\rho)=F(\mu,Ku(\Omega))$, hence $F(j,Ku(\Omega))$ is contained in $F(\mu,\rho)$. Therefore we get the inclusion relation $F(j,Ku(\Omega))\subset F(\mu,\rho)\subset F(j,\Omega)$, hence by putting $k_0=F(\rho)$ we get $k_0(j,Ku(\Omega))\subset k_0(\mu)\subset k_0(j,\Omega)$. However $[k_0(j,\Omega):k_0(j,Ku(\Omega))]$ is at most equal to 2 and the Galois group of $k_0(\mu)$ over $k_0(j)$, which is the alternating group of permutations of four letters, does not contain any normal subgroup of order 2. Therefore we have $k_0(j,Ku(\Omega))=k_0(\mu)$, hence $k(j,Ku(\Omega))=k(\mu)$.

Now we shall examine the extensions $k(\lambda)$ and $k(\mu)$ of $k(j)$ more closely for our later purpose. Consider $k(\lambda)$ first assuming that the characteristic is different from 2. We know that $k(\lambda)$ is ramified over $k(j)$ at $j=0,12^3,\infty$ and nowhere else. Moreover in case the characteristic is different from 3 the ramification is tame and the corresponding inertia groups are cyclic of order 3, 2, 2. If the characteristic is 3, then $k(\lambda)$ is ramified only at $j=0$ and ∞. It is still tamely ramified at $j=\infty$ and the corresponding inertia groups are cyclic of order 2. However at $j=0$ the ramification is wild. There is only one point $\lambda=-1$ above $j=0$, hence the inertia group is the whole Galois group. Moreover, as we can see, the cyclic subgroup of order 3, necessarily consisting of transformations $\lambda\rightarrow\lambda$, $(1-\lambda)^{-1}$, $(\lambda-1)\lambda^{-1}$, is the first ramification group while the second ramification group reduces to the identity. The different of the covering is, therefore, given by $(0)+(1)+(\infty)+7.(-1)$. This agrees with the expression $dj=(1+\lambda)^7\lambda^{-3}(1-\lambda)^{-3}d\lambda$. Next consider $k(\mu)$ assuming that the characteristic is different from 3. We know that $k(\mu)$ is ramified over $k(j)$ at $j=0,12^3,\infty$ and nowhere else. Moreover in case the characteristic is different from 2 the ramification is tame and the corresponding inertia groups are cyclic of order 3, 2, 3. If the characteristic is 2, then $k(\mu)$ is ramified only at $j=0$ and ∞. It is still tamely ramified at $j=\infty$ and the corresponding inertia groups are cyclic of order 3. However at $j=0$ the ramification is wild. There is only one point $\mu=0$ above $j=0$, hence the inertia group is the whole Galois group. Moreover, as we can see, the Klein four group, necessarily consisting of transformations $\mu\rightarrow\mu$, $\zeta\mu(\mu-\zeta)^{-1}$ for $\zeta=1,\rho,\rho'$, is the first ramification group while the second ramification group reduces to the identity. The different of the covering is,

therefore, given by $2.((1)+(\rho)+(\rho')+(\infty))+14.(0)$. This agrees again with the expression $dj=\mu^{14}(\mu^3-1)^{-4}d\mu$.

Finally we note that every tamely ramified extension of $k(j)$ ramified only at two points is cyclic. This follows at once from the *relative genus formula*, which is as follows: Let Σ be a finite covering of a curve K. Let g and g_0 be the genera of Σ and K; let $\mathfrak{d}$ be the different of the covering. Then we have $2g-2=\deg(\mathfrak{d})+[\Sigma:K](2g_0-2)$. On the other hand, as it was observed by Abhyankar [1], the field $k(j)$ can be covered by fields of "large Galois groups" ramified only at one point.

4. Degeneration and ramification. There is a close connection between degenerations of elliptic curves and ramifications of certain extensions of parameter fields. The examination of this connection will provide *key lemmas* for the Galois theory of points of finite orders. We shall start by examining degenerate members of the linear pencils of elliptic curves introduced in Section 2. Let F be, as before, a prime field of an arbitrary characteristic. In case the characteristic is different from 2 we consider the linear pencil defined over F by $Y^2Z=X(X-Z)(X-\lambda Z)$. This linear pencil contains three singular members which correspond to $\lambda=0,1,\infty$. The singular member which corresponds to $\lambda=\infty$ is a reducible cubic. However two other singular members are irreducible and the singular points $(0,0,1)$ for $\lambda=0$ and $(1,0,1)$ for $\lambda=1$ are what we call *ordinary double points*. In case the characteristic is different from 3 we consider the linear pencils defined over F by $X^3+Y^3+Z^3=3\mu XYZ$ and $Y^2Z-YZ^2=X^3+3\nu XYZ$. Both linear pencils contain four singular members which correspond to $\mu,\nu=1,\rho,\rho',\infty$. The singular members which correspond to $\mu=1,\rho,\rho',\infty$ and $\nu=\infty$ are all reducible cubics. However singular members which correspond to $\nu=1,\rho,\rho'$ are irreducible and the respective singular points $(1,1,-1)$, $(\rho',1,-1)$, $(\rho,1,-1)$ are again ordinary double points. In this connection we remark also that $Y^2Z=4X^3-3^3j(j-12^3)^{-1}(X+Z)Z^3$ reduces at $j=\infty$ to an irreducible cubic with an ordinary double point $(-\frac{3}{2},0,1)$. Similarly $Y^2Z=X^3-X^2Z+j^{-1}Z^3$ reduces at $j=\infty$ to an irreducible cubic with an ordinary double point at $(0,0,1)$. However $Y^2Z-XYZ=j^{-1}X^3+jZ^3$ will become reducible at $j=\infty$.

Consider in general an elliptic curve defined over a field K which is complete under a real discrete valuation. Let k be the residue field and let A' be the unique specialization of A over the natural homomorphism of the valuation ring of K onto k. We assume that A' is either non-singular or has one ordinary double point different from the image of the neutral element

of A. We also assume that k is algebraically closed. Let n be a natural number not divisible by the characteristic of k and let Ω be the group of points of A of order n. Then, *in case A' is non-singular we have $K(\Omega) = K$. In case A' has an ordinary double point Ω contains a base ω_1, ω_2 on which the Galois group of $K(\Omega)$ over K operates as follows*

$$\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix} \to \begin{pmatrix}1 & m\\ 0 & 1\end{pmatrix}\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}.$$

The cyclic group or order n generated by ω_2, which we called the *group of vanishing points of order n*, is uniquely determined by A. Here m is an integer modulo n. This we proved separately in a more general case of Jacobian varieties [7, 9].

Now, as before, take k to be the algebraic closure of F. Let j be a variable over k and consider A_j. Then, *no matter how we construct A_j, this must degenerate at least at $j = 0$, 12^3 and ∞.* In fact, suppose that the characteristic is different from 2 and let Ω be the group of points of A_j of order 2. If (a) is a rational point of a projective straight line over k, we take the power-series field $k((j-a))$ as K and A_j as A. In this way, if A_a is non-singular, we see that $k(j, Ku(\Omega)) = k(\lambda)$ is unramified over $k(j)$ at $j = a$. However we know that $k(\lambda)$ is ramified over $k(j)$ at $j = 0$, 12^3 and ∞. Therefore A_j can not remain to be non-singular at these points. If the characteristic is 2, we consider the group Ω of points of A_j of order 3 and apply the fact that $k(j, Ku(\Omega)) = k(\mu)$ is ramified over $k(j)$ at $j = 0$ and ∞. On the other hand, let Ω_λ, Ω_μ and Ω_ν be the groups of points of order n on the elliptic curves $Y^2Z = X(X-Z)(X-\lambda Z)$, $X^3 + Y^3 + Z^3 = 3\mu XYZ$ and $Y^2Z - YZ^2 = X^3 + 3\nu XYZ$. Then by a similar consideration we see that $k(\lambda, \Omega_\lambda)$ is ramified over $k(\lambda)$ only at $\lambda = 0, 1, \infty$ and the ramification is tame at $\lambda = 0, 1$. We see also that $k(\mu, \Omega_\mu)$ and $k(\nu, \Omega_\nu)$ are ramified over $k(\mu)$ and $k(\nu)$ respectively only at $\mu, \nu = 1, \rho, \rho', \infty$ and $k(\nu, \Omega_\nu)$ is tamely ramified at $\nu = 1, \rho, \rho'$. Moreover in case the ramification is tame the corresponding inertia groups are cyclic of an order which divides n. These local informations will play a definitive role in determining the Galois groups explicitly. There are some more things to be cleared up before we start discussing those Galois groups.

One thing to be remarked is that *$Y^2Z = X(X-Z)(X-\lambda Z)$ has only four rational points over $k(\lambda)$, and these are exactly points of order 2.* In fact, let v be a rational point of $Y^2Z = X(X-Z)(X-\lambda Z)$ over $k(\lambda)$ different from the base points $(0,1,0)$, $(0,0,1)$, $(1,0,1)$ and consider its locus C over k. Then $Y^2Z = X(X-Z)(X-\lambda Z)$ intersects with C at v

with unit multiplicity and extra intersections are among the three base points, which together with $v_0=(\lambda,0,1)$ form the group of points of order 2. In fact the restriction of the linear pencil to C gives rise to a linear pencil on C. The "variable part" of a linear pencil is, in general, a prime rational divisor over the parameter field [cf. 17]. However, since v is a member of this variable part and since v is rational over $k(\lambda)$, we see that v itself is the variable part. It is clear that the "fixed part" consists of the above mentioned base points of the original linear pencil. Therefore, if we denote the X-axis by C_0, then $Y^2Z=X(X-Z)(X-\lambda Z)$ intersects with $C-\deg(C)\cdot C_0$ at $(v)-\deg(C)\cdot(v_0)$ modulo base points. However, since $C-\deg(C)\cdot C_0$ is a divisor of a function on the projective plane, by applying the so-called Abel theorem we see that v is a point of order 2. The only possibility is then $v=v_0$. This proves the assertion. In the same way we see that $X^3+Y^3+Z^3=3\mu XYZ$ *has nine rational points over* $k(\mu)$, *and these are exactly points of order* 3. Moreover $Y^2Z-YZ^2=X^3+3\nu XYZ$ *has only three rational points over* $k(\nu)$, *and these are* $(0,1,0)$, $(0,1,1)$, $(0,0,1)$.

Another thing to be remarked is the following: We know that every elliptic curve, in particular A_j, can be transformed into the normal forms discussed in Section 2. We shall investigate smallest fields over which these transformations are possible. Assume first that the characteristic is different from 2. Then we can transform A_j to $\alpha\cdot Y^2Z=X(X-Z)(X-\lambda Z)$ over $F(\lambda)$. This α is not uniquely determined by A_j. At any rate, in order to get $Y^2Z=X(X-Z)(X-\lambda Z)$, we must introduce $\alpha^{\frac{1}{2}}$, i.e. we must possibly go through a quadratic extension of $F(\lambda)$. On the other hand the discussion in Section 2 shows that the six conjugates of $Y^2Z=X(X-Z)(X-\lambda Z)$ over $F(j)$ can not be transformed into each other over $F(\lambda)$. However, if we introduce κ and κ' by $\kappa^2=\lambda$ and $(\kappa')^2=1-\lambda$, then the twelve isomorphisms are all defined over $k_0(\kappa,\kappa')$. Here k_0 is the field of fourth roots of unity. The field $k_0(\kappa,\kappa')$ has another meaning. If x and y are the inhomogeneous co-ordinate functions of $Y^2Z=X(X-Z)(X-\lambda Z)$, the *addition theorem* of x can be obtained by an elementary analytic geometry and we get

$$x(u+v)=x(u)x(v)(\lambda-x(u)x(v))^2(x(u)y(v)+x(v)y(u))^{-2}.$$

Therefore, if Ω_λ is the group of points of $Y^2Z=X(X-Z)(X-\lambda Z)$ of order 4, we have $F(\lambda,\Omega_\lambda)=k_0(\kappa,\kappa')$. At any rate, if we fix A_j, although $\alpha^{\frac{1}{2}}$ is not unique, it determines a unique extension of $k_0(\kappa,\kappa')$. We know that α is an element of $F(\lambda)$. If λ' corresponds to j' at which A_j remains to be nonsingular, certainly $\alpha(\lambda')$ is different from 0 and ∞. Now, if the characteristic is also different from 3 and if we take $Y^2Z=4X^3-3^3j(j-12^3)^{-1}(X+Z)Z^2$

as A_j, then A_j degenerates only at $j=0$, 12^3 and ∞. Therefore

$$((1-\lambda+\lambda^2)(1+\lambda)(1-2\lambda)(2-\lambda))^{\frac{1}{2}}$$

is one of the possible radicals to be introduced over $k_0(\kappa,\kappa')$ and we can show that this is exactly the radical we need. If the characteristic is 3 and if we take $Y^2Z=X^3-X^2Z+j^{-1}Z^3$ as A_j, then A_j degenerates only at $j=0$ and ∞. Therefore $(1+\lambda)^{\frac{1}{2}}$ is one of the possible radicals to be introduced over $k_0(\kappa,\kappa')$ and we can show again that this is the radical we need. Next, assume that the characteristic is different from 3. Let Ω be the group of points of A_j of order 3. Then we can transform A_j to $X^3+Y^3+Z^3=3\mu XYZ$ over $F(j,\Omega)$. In this case the twelve conjugates of $X^3+Y^3+Z^3=3\mu XYZ$ over $F(j)$ can be transformed into each other over $k_0(\mu)$. Here k_0 is now the field of third roots of unity. We know that $k_0(j,\Omega)=F(j,\Omega)$ is at most a quadratic extension of $k_0(\mu)$. If the characteristic is also different from 2 and if we take $Y^2Z=4X^3-3^3j(j-12^3)^{-1}(X+Z)Z^2$ as A_j, we can see that $(2\mu(8+\mu^3)(8+20\mu^3-\mu^6))^{\frac{1}{2}}$ is the radical to be introduced over $k_0(\mu)$. In other words $k_0(j,\Omega)$ is indeed a quadratic extension of $k_0(\mu)$ obtained by adjoining the above radical to it. If the characteristic is 2 and if we take $Y^2Z-XYZ=j^{-1}X^3+jZ^3$ as A_j, we see that $k_0(j,\Omega)$ is the quadratic extension of $k_0(\mu)$ obtained by adjoining a root θ of $\theta^2-\theta=\mu^{-3}$ to it. We note that $k_0(j,\Omega)$ is, in this case, ramified over $k_0(\mu)$ only at $\mu=0$. Moreover the inhomogeneous equation $y^2-y=x^3$ defines an elliptic curve in characteristic 2 with $j=0$ as its absolute invariant.

In connection with what we discussed above we remark also the following. Suppose that μ and ν correspond to the same j which we assume to be a variable over the field k_0 of the third roots of unity. Then the transformations between $X^3+Y^3+Z^3=3\mu XYZ$ and $Y^2Z-YZ^2=X^3+3\nu XYZ$ are defined over $k_0(\mu,\nu)$. In fact, since the twelve conjugates of $X^3+Y^3+Z^3=3\mu XYZ$ over $F(j)$ are transformed into each other over $k_0(\mu)$, hence over $k_0(\mu,\nu)$, we have only to show that a transformation is possible over $k_0(\mu,\nu)$ for some choice of μ and ν. However this we have shown explicitly in Section 2. The field $k_0(\mu,\nu)$ has another meaning. If Ω_ν is the group of points of $Y^2Z-YZ^2=X^3+3\nu XYZ$ of order 3, we have $F(\nu,\Omega_\nu)=k_0(\mu,\nu)$. At any rate $F(\nu,\Omega_\nu)=k_0(\nu,\Omega_\nu)$ is contained in $k_0(\mu,\nu)$ by what we have remarked above. On the other hand we know that only three members of Ω_ν are rational over $k_0(\nu)$ and $k_0(\mu,\nu)$ is of degree 3 over $k_0(\nu)$. Hence $F(\nu,\Omega_\nu)$ $=k_0(\mu,\nu)$ is the only possibility we have.

5. Galois theory of division points. We are now completely prepared

to attack our main problem.[3] Suppose that F is, as before, a prime field of an arbitrary characteristic and k its algebraic closure. The moduli λ, μ, ν are assumed to correspond to the same j which is a variable over k. Let n be a natural number not divisible by the characteristic of F. Then the groups of points of order n on A_j, $Y^2Z = X(X-Z)(X-\lambda Z)$, $X^3+Y^3+Z^3 = 3\mu XYZ$ and $Y^2Z - YZ^2 = X^3 + 3\nu XYZ$ will be denoted by Ω, Ω_λ, Ω_μ and Ω_ν. Also points of exact order n will be called primitive n-th division points.

We shall first discuss the case where the characteristic is different from 2. Then we have $k(\lambda, Ku(\Omega_\lambda)) = k(\lambda, Ku(\Omega))$ and both $k(\lambda)$ and $k(j, Ku(\Omega))$ are normal over $k(j)$, hence $k(\lambda, Ku(\Omega_\lambda))$ is normal over $k(j)$. On the other hand we know that $k(\lambda, \Omega_\lambda)$ is ramified over $k(\lambda)$ only at $\lambda = 0$, 1 and ∞. Since these points are conjugate over $k(j)$, we conclude that the inertia groups of $k(\lambda, Ku(\Omega_\lambda))$ over $k(\lambda)$ are conjugate in the Galois group of $k(\lambda, Ku(\Omega_\lambda))$ over $k(j)$. We also know that the orders of inertia groups of $k(\lambda, \Omega_\lambda)$ over $k(\lambda)$ at $\lambda = 0$ and $\lambda = 1$ divide n. Therefore the order of the inertia groups of $k(\lambda, \Omega_\lambda)$ over $k(\lambda)$ at $\lambda = \infty$ is, at any rate, a divisor of $2n$. Hence $k(\lambda, \Omega_\lambda)$ is tamely ramified over $k(\lambda)$. As a consequence, if n splits into relatively prime factors n' and n'', the field $k(\lambda, n''\cdot\Omega_\lambda)$ of n'-th division points and the field $k(\lambda, n'\cdot\Omega_\lambda)$ of n''-th division points are linearly disjoint over $k(\lambda)$. Otherwise their intersection Σ will be a proper Galois extension of $k(\lambda)$ which is tamely ramified over $k(\lambda)$. Moreover Σ can be ramified over $k(\lambda)$ only at $\lambda = \infty$. This is a contradiction.

Let therefore n be a power of a prime number p. Assume first that p is odd. Let ω be a primitive n-th division point in Ω_λ. Then we can find an inertia group of $k(\lambda, \Omega_\lambda)$ over $k(\lambda)$ at $\lambda = 0$ or at $\lambda = 1$ under which ω is transformed into n distinct conjugates. Otherwise $n/p\cdot\omega$ will be kept invariant by all inertia groups of $k(\lambda, \Omega_\lambda)$ over $k(\lambda)$ at $\lambda = 0$ and $\lambda = 1$, i.e. the subextension $k(\lambda, n/p\cdot\omega)$ of $k(\lambda, \Omega_\lambda)$ will be unramified over $k(\lambda)$ at $\lambda = 0$ and $\lambda = 1$. This implies that $n/p\cdot\omega$ is rational over $k(\lambda)$. However this is not possible, because p is odd. Therefore we can find an inertia group of $k(\lambda, \Omega_\lambda)$ over $k(\lambda)$ and the corresponding base ω_1, ω_2 of Ω_λ such that the inertia group operates on this base as $(\omega_1, \omega_2) \to (\omega_1 + m\omega_2, \omega_2)$ for $m = 1, \cdots$. In the same way we can find another inertia group of $k(\lambda, \Omega_\lambda)$ over $k(\lambda)$ which operates as $(\omega_2, \omega_1 + c\omega_2) \to (\omega_2 + m(\omega_1 + c\omega_2), \omega_1 + c\omega_2)$ for $m = 1, \cdots$. Here, by replacing ω_1 by $\omega_1 - c\omega_2$, we can assume that we have $c = 0$ from the beginning. Then the Galois group of $k(\lambda, \Omega_\lambda)$ over $k(\lambda)$ contains two

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[3] **If the reader is interested only in the field of modular functions with level, he can skip over some preparations and also some argument in this section.**

substitutions of the form $(1\ 1, 0\ 1)$ and $(1\ 0, 1\ 1)$ both modulo n. However these substitutions generate the whole $SL(2,n)$. Therefore, by the previous remark, if n is just an odd natural number, the Galois group of $k(\lambda, \Omega_\lambda)$ over $k(\lambda)$ is $SL(2,n)$.[4] Consequently the Galois group of $k(\lambda, Ku(\Omega_\lambda))$ over $k(\lambda)$ is $LF(2,n)$. Since, in general the Galois group of $k(j, Ku(\Omega))$ over $k(j)$ is at most $LF(2,n)$, this group is also $LF(2,n)$. In particular $k(\lambda)$ and $k(j, Ku(\Omega))$ are linearly disjoint over $k(j)$. Moreover the Galois group of $k(j,\Omega)$ over $k(j)$ must be $SL(2,n)$. Otherwise we have $k(j,\Omega) = k(j, Ku(\Omega))$, hence $k(\lambda, \Omega_\lambda)$ is a quadratic extension of $k(\lambda, Ku(\Omega_\lambda)) = k(\lambda, Ku(\Omega)) = k(\lambda, \Omega)$. Therefore the quadratic extension of $k(\lambda)$ over which A_j can be transformed into $Y^2Z = X(X-Z)(X-\lambda Z)$ is contained in $k(\lambda, \Omega_\lambda)$, hence $k(\lambda, \Omega_\lambda)$ will be the compositum of $k(\lambda, Ku(\Omega_\lambda))$ and this quadratic extension over $k(\lambda)$. This is a contradiction, because $SL(2,n)$ does not split into the direct product of $LF(2,n)$ and Z.

Next, consider the case $p=2$. As before, we can find a base ω_1, ω_2 of Ω_λ such that one inertia group of $k(\lambda, \Omega_\lambda)$ over $k(\lambda)$ operates on this base as $(\omega_1, \omega_2) \to (\omega_1 + 2m\omega_2, \omega_2)$ for $m = 1, \cdots$ and another inertia group operates as $(\omega_2, \omega_1) \to (\omega_2 + 2m\omega_1, \omega_1)$ for $m = 1, \cdots$. Then the Galois group of $k(\lambda, \Omega_\lambda)$ over $k(\lambda)$ contains two substitutions of the form $(1\ 2, 0\ 1)$ and $(1\ 0, 2\ 1)$ both modulo n. Hence the Galois group of $k(\lambda, Ku(\Omega_\lambda))$ over $k(\lambda)$ contains the normal subgroup of $LF(2,n)$ of index 6 whose elements are represented by $(a,b,c,d) \equiv (1\ 0, 0\ 1) \bmod 2$. Therefore, because of the relation $k(j, Ku(\Omega)) = k(\lambda, Ku(\Omega))$, the Galois group of $k(\lambda, Ku(\Omega_\lambda)) = k(\lambda, Ku(\Omega))$ over $k(\lambda)$ is exactly that group and the Galois group of $k(j, Ku(\Omega))$ over $k(j)$ is the whole $LF(2,n)$. On the other hand, in this case, we have $k(\lambda, \Omega_\lambda) = k(\lambda, Ku(\Omega_\lambda))$. Otherwise n is at least equal to 4 and the Galois group of $k(\lambda, \Omega_\lambda)$ over $k(\lambda)$ will contain the substitution which transforms every element of Ω_λ to its inverse. The same situation must prevail for $n/4 \cdot \Omega_\lambda$. However we know that the field $k(\lambda, n/4 \cdot \Omega_\lambda)$ of fourth division points is an extension of $k(\lambda)$ of degree 4, hence we get $k(\lambda, n/4 \cdot \Omega_\lambda) = k(\lambda, Ku(n/4 \cdot \Omega_\lambda))$. This is a contradiction, hence the assertion is proved. However, if we exclude the case $n=2$, then $k(j,\Omega)$ is a quadratic extension of $k(j, Ku(\Omega))$. In fact, let Σ be the quadratic extension of the field of fourth division points over which A_j can be transformed into $Y^2Z = X(X-Z)(X-\lambda Z)$. Then $\Sigma(\Omega_\lambda) = \Sigma(\Omega)$ is a quadratic extension of $k(\lambda, \Omega_\lambda)$, as it is clear from the consideration in the previous section, and its non-identical automorphism transforms every element of Ω to its inverse. This proves the assertion.

[4] The original argument of the author was a little bit more complicated. A simplification was made through a discussion with Professor Tamagawa in the Fall of 1957.

Since we know that the Galois group of $k(j, Ku(\Omega))$ over $k(j)$ is $LF(2,n)$, the Galois group of $k(j,\Omega)$ over $k(j)$ is $SL(2,n)$. We can summarize some of our results as follows:[5]

THEOREM 1. *Let λ be transcendental over a prime field F of characteristic different from 2. Let n be a natural number not divisible by the characteristic and let Ω_λ be the group of points of $Y^2Z = X(X-Z)(X-\lambda Z)$ of order n. Then $F(\lambda,\Omega_\lambda)$ contains the field k_0 of n-th roots of unity and k_0 is algebraically closed in $k_0(\lambda,\Omega_\lambda)$. Moreover the Galois group of $k_0(\lambda,\Omega_\lambda)$ over $k_0(\lambda)$ is the subgroup of $SL(2,n)$ defined by*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \bmod 4.$$

It might not be necessary, but, to make sure, we shall explain why the theorem for k_0 follows from the above consideration for k. We know that $F(\lambda,\Omega_\lambda)$ contains k_0. Also we can see quite easily that the Galois group of $k_0(\lambda,\Omega_\lambda)$ over $k_0(\lambda)$ is, at any rate, a subgroup of the group in the theorem, which is the Galois group of $k(\lambda,\Omega_\lambda)$ over $k(\lambda)$. Therefore these two Galois groups must be the same. In other words $k_0(\lambda,\Omega_\lambda)$ and $k(\lambda)$ are linearly disjoint over $k_0(\lambda)$, hence $k_0(\lambda,\Omega_\lambda)$ is a regular extension of k_0.

Next we shall consider the case of characteristic 2 or, more generally, the case where the characteristic is different from 3. We can assume that the moduli μ and ν are related by $(\mu^3-1)(\nu^3-1)=1$. Then $k(\mu)$ and $k(\nu)$ are linearly disjoint cyclic extensions of degree 3 over $k(\mu^3)=k(\nu^3)$. We shall show that $k(\mu,\Omega_\mu)$ and $k(\nu,\Omega_\nu)$ are tamely ramified over $k(\mu)$ and over $k(\nu)$. We know that $k(\nu,\Omega_\nu)$ is tamely ramified over $k(\nu)$ at $\nu=1,\rho,\rho'$. Therefore $k(\mu,\nu,\Omega_\mu)=k(\mu,\nu,\Omega_\nu)$ is tamely ramified over $k(\mu,\nu)$ at $\mu=\infty$. Since the characteristic is different from 3, it is also tamely ramified over $k(\mu)$ at $\mu=\infty$. Since the twelve conjugates of $X^3+Y^3+Z^3=3\mu XYZ$ over $k(j)$ are transformable into each other over $k(\mu)$, we conclude that $k(\mu,\Omega_\mu)$ is normal over $k(j)$. Since the Galois group of $k(\mu,\Omega_\mu)$ over $k(j)$ operates transitively on $\mu=1,\rho,\rho',\infty$, the inertia groups of $k(\mu,\Omega_\mu)$ over $k(\mu)$ are conjugate in this Galois group. In particular $k(\mu,\Omega_\mu)$ is tamely ramified over $k(\mu)$. Hence going backwards we see that $k(\nu,\Omega_\nu)$ is also tamely ramified over $k(\nu)$ as asserted. Since the orders of the inertia groups of $k(\nu,\Omega_\nu)$ over $k(\nu)$ at $\nu=1,\rho,\rho'$ divide n, if n splits into relatively prime factors n' and n'', the field $k(\nu,n''\cdot\Omega_\nu)$ of n'-th division points and the field $k(\nu,n'\cdot\Omega_\nu)$ of n''-th

[5] The special case of this theorem in the case of characteristic zero and for odd n was proved by Kronecker using general transformation formulas of Gauss-Jacobi theta functions [12, 13]. For a systematic treatment, see [15].

division points are linearly disjoint over $k(\nu)$. Moreover, if we compare $k(\mu,\Omega_\mu)$ with $k(\nu,\Omega_\nu)$ over $k(\mu^3)=k(\nu^3)$ at $\mu=\infty$, we can see that the order of the inertia groups of $k(\mu,\Omega_\mu)$ over $k(\mu)$ at $\mu=\infty$, hence also at $\mu=1,\rho,\rho'$, divide n. Therefore, as above, the splitting of n into relatively prime factors results in the splitting of $k(\mu,\Omega_\mu)$ into linearly disjoint subextensions over $k(\mu)$.

Let therefore n be a power of a prime number p. Assume first that p is different from 3. Then, as before, we can conclude that the Galois groups of $k(\mu,\Omega_\mu)$ and $k(\nu,\Omega_\nu)$ over $k(\mu)$ and $k(\nu)$ are isomorphic to $SL(2,n)$. The same situation prevails provided n is not divisible by 3. This implies that the Galois group of $k(j,Ku(\Omega))$ over $k(j)$ is $LF(2,n)$ and, as before, that the Galois group of $k(j,\Omega)$ over $k(j)$ is $SL(2,n)$. Next, consider the case $p=3$. Then, since it is so for $n=3$, we have $k(\mu,\Omega_\mu)=k(\mu,Ku(\Omega_\mu))$ in general. Moreover, by a similar consideration as before, we see that the Galois group of $k(\mu,\nu,\Omega_\mu)=k(\mu,\nu,\Omega_\nu)$ over $k(\mu,\nu)$ contains two substitutions of the form $(1\ 3,0\ 1)$ and $(1\ 0,3\ 1)$ both modulo n. Therefore the Galois group of $k(\mu,\Omega)=k(j,\Omega)$ over $k(j)$ also contains these substitutions, hence it must be the whole $SL(2,n)$. Consequently the Galois group of $k(j,Ku(\Omega))$ over $k(j)$ is $LF(2,n)$. This implies immediately that the Galois group of $k(\mu,\Omega_\mu)$ over $k(\mu)$ is the normal subgroup of $SL(2,n)$ of index 24 defined by $(a\ b,c\ d)\equiv(1\ 0,0\ 1)$ mod 3. We can summarize some of our results as follows:

THEOREM 2. *Let μ be transcendental over a prime field F of characteristic different from 3. Let n be a natural number not divisible by the characteristic and let Ω_μ be the group of points of $X^3+Y^3+Z^3=3\mu XYZ$ of order n. Then $F(\mu,\Omega_\mu)$ contains the field k_0 of n-th roots of unity and k_0 is algebraically closed in $k_0(\mu,\Omega_\mu)$. Moreover the Galois group of $k_0(\mu,\Omega_\mu)$ over $k_0(\mu)$ is the subgroup of $SL(2,n)$ defined* by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod 3.$$

Moreover, if we understand by A_j the plane cubics introduced in Section 3, we can state the following theorem:

THEOREM 3. *Let j be transcendental over a prime field F of an arbitrary characteristic. Let n be a natural number not divisible by the characteristic and let Ω be the group of points of A_j of order n. Then $F(j,\Omega)$ contains the field k_0 of n-th roots of unity and k_0 is algebraically closed in $k_0(j,\Omega)$. Moreover the Galois group of $k_0(j,\Omega)$ over $k_0(j)$ is $SL(2,n)$ and the Galois*

group of $F(j,\Omega)$ *over* $F(j)$ *consists of elements of* $GL(2,n)$ *whose determinants give rise to the well-known automorphisms of* k_0 *over* F.

The last statement requires some explanation. Any automorphism of $F(j,\Omega)$ over $F(j)$ can be represented, with reference to a base of Ω, by an element $(a\ b, c\ d)$ of $GL(2,n)$. The theorem asserts that this automorphism transforms each n-th root of unity into its $(ad-bc)$-th power. If the characteristic is zero, there is no condition. However, if the characteristic is positive, this means that $ad-bc$ is congruent to a power of the characteristic modulo n. At any rate, if an element of $GL(2,n)$ satisfies this condition, it appears as an automorphism of $F(j,\Omega)$ over $F(j)$. We note that exactly the same statement can also be added to Theorem 1 and Theorem 2.

On the other hand, if we fix a prime number p different from the characteristic of F and if we denote by Ω the p-primary part of the group of points of A_j of finite orders, the algebraic closure k_0 of F in $F(j,\Omega)$ is the field of all p^e-th roots of unity for $e=1,\cdots$. Moreover the Galois group of $k_0(j,\Omega)$ over $k_0(j)$ with Krull's topology is the special linear group of two-by-two matrices over the ring of Hensel's p-adic integers.

6. Fields of modular functions. The field of modular functions of level n was introduced in Section 3 as follows: Let k be the algebraic closure of the prime field F of characteristic not dividing n. Let j be transcendental over k and let Ω be the group of points of A_j of order n. Then $k(j,Ku(\Omega))$ is the field in question. In the previous section we determined the Galois group of $k(j,Ku(\Omega))$ over $k(j)$, i.e. we proved the following theorem:

THEOREM 4. *The Galois group of the field of modular functions of level* n *over* $k(j)$ *is* $LF(2,n)$.

For $n=3$ the Galois group is called the *tetrahedral group.* As we know, the tetrahedral group is just the alternating group of permutations of four letters. Similarly for $n=5$ the Galois group is called the *icosahedral group.* This is the first simple group in this sequence of groups and it is the alternating group of permutations of five letters. As we shall see presently, the genus of the field $k(j,Ku(\Omega))$ is zero up to $n=5$ and hence, in the case of characteristic zero, the Galois group is represented as a group of rotations of a sphere, which gives rise to a polyhedral decomposition of the sphere. This is why the Galois groups have such geometric names.

THEOREM 5. *The genus* g *of the field of modular functions of level* n *is given by*

$$2g-2=1/12\cdot(n-6)\phi(n)\psi(n).$$

Here the case $n=2$ is exceptional and, in that case, we just get $g=0$.

First we note that the ramification index of $k(j,Ku(\Omega))$ over $k(j)$ at $j=\infty$ is n. In case the characteristic is different from 2, by the remark in Section 4, we know that the ramification index is a divisor of n. Suppose that n is even. Then $k(j,Ku(\Omega))$ contains $k(\lambda)$ and the ramification index of $k(\lambda,Ku(\Omega))$ over $k(\lambda)$ at $\lambda=0$, say, is $n/2$ while the ramification index of $k(\lambda)$ over $k(j)$ at $j=\infty$ is 2, whence the assertion. Suppose next that n is odd. Then the ramification index of $k(j,Ku(\Omega))$ over $k(j)$ at $j=\infty$ is at least equal to n, whence it must be equal to n. In the case of characteristic 2 we can conclude as follows. We know that the ramification index of $k(\nu,Ku(\Omega))$ over $k(\nu)$ at $\nu=1$, say, is n while $\nu=1$ covers $j=\infty$ only once. Therefore the ramification index of $k(j,Ku(\Omega))$ over $k(j)$ at $j=\infty$ must be n. This completes the proof. As a consequence the ramification is always tame at $j=\infty$. Next we shall determine other ramifications. Suppose that the characteristic is different from 2. Then $k(\lambda,Ku(\Omega))$ is ramified over $k(j)$ only at $j=0$, 12^3 and ∞, hence $k(j,Ku(\Omega))$ can be ramified over $k(j)$, besides $j=\infty$, only at $j=0$ and $j=12^3$. Moreover, if the characteristic is also different from 3, the ramification indices of $k(\lambda,Ku(\Omega))$ over $k(j)$ at $j=0$ and $j=12^3$ are 3 and 2. Therefore the ramification indices of $k(j,Ku(\Omega))$ over $k(j)$ at $j=0$ and $j=12^3$ divide 3 and 2. In particular $k(j,Ku(\Omega))$ is tamely ramified over $k(j)$. Therefore, if we remark that the Galois group is non-commutative, the ramification must take place at least at three points. Hence $k(j,Ku(\Omega))$ is ramified over $k(j)$ at $j=0$ and $j=12^3$ with respective ramification indices 3 and 2. Then by computing the degree of the divisor (dj) in $k(j,Ku(\Omega))$ we get the genus formula in this case. If the characteristic is 3, the ramification takes place only at $j=0$ and $j=\infty$, hence the ramification is wild at $j=0$. We shall show that the ramification index at $j=0$ is 6. We know that the inertia group of $k(\lambda,Ku(\Omega))$ over $k(j)$ at $j=0$ is the same as the inertia group of $k(\lambda)$ over $k(j)$ at $j=0$. Since this group contains only one non-trivial normal subgroup, which is of index 2, if the ramification index is not 6, it must be 2. Then the ramification is tame at $j=0$, but this is a contradiction. Once we know that the inertia group of $k(j,Ku(\Omega))$ over $k(j)$ at $j=0$ is the same as the inertia group of $k(\lambda)$ over $k(j)$ at $j=0$, we get the same genus formula as before. Finally, the case of characteristic 2 can be treated similarly through the field $k(\mu,Ku(\Omega))$. We see that $k(j,Ku(\Omega))$ is ramified over $k(j)$ only at $j=0$ and $j=\infty$, the ramification being wild at

$j=0$. We know that the inertia group of $k(\mu, Ku(\Omega))$ over $k(j)$ at $j=0$ is the same as the inertia group of $k(\mu)$ over $k(j)$ at $j=0$. Since this group contains only one non-trivial normal subgroup, which is of index 3, if the ramification index is not 12, it must be 3. Then the ramification is tame at $j=0$, and we get a contradiction. Therefore the inertia group of $k(j, Ku(\Omega))$ over $k(j)$ at $j=0$ must be the same as the inertia group of $k(\mu)$ over $k(j)$ at $j=0$. Hence again we get exactly the same genus formula.

In the course of the proof of Theorem 5 we also proved the following theorem:

THEOREM 6. *The field of modular functions of level n is ramified over $k(j)$ only at $j=0$, 12^3 and ∞. The ramification is always tame at $j=\infty$ with n as its index. If the characteristic is different from 2 and 3, it is tamely ramified at $j=0$ and $j=12^3$ with respective indices 3 and 2. If the characteristic is 3, it is wildly ramified at $j=0$ with the symmetric group of permutations of three letters as its inertia group and with the alternating group as the first ramification group; the second ramification group reduces to the identity. If the characteristic is 2, it is wildly ramified at $j=0$ with the tetrahedral group as its inertia group and with the Klein four group as the first ramification group; the second ramification group reduces to the identity.*

The Theorem 4 and the Theorem 6 are fundamental in the theory of modular functions with level. We can, for instance, calculate genera of all fields in between $k(j, Ku(\Omega))$ and $k(j)$ with the aid of *Hilbert's Galois theory* [6]. We shall illustrate this calculation by a special but important case of the *field of invariant transformation equation* of degree n. This field, say Σ, is defined to be the field which corresponds to the subgroup, say Π, of $LF(2,n)$ whose elements are represented by $(a\ b, c\ d)$ with $c\equiv 0 \bmod n$. We note that this field is of degree $\psi(n)$ over $k(j)$ and it is determined up to a conjugacy in $k(j, Ku(\Omega))$.

THEOREM 7. *The genus g of the field of invariant transformation equation of degree n is given by*[6]

$$2g-2=1/6\cdot\psi(n)-1/2\cdot\pi_2(n)-2/3\cdot\pi_3(n)-\sum_{mm'=n}\phi((m,m')).$$

[6] In the case of characteristic zero and for a prime level p we find this formula already in Klein [10]. We note that $g+1$ is then the number of non-isomorphic elliptic curves in characteristic p having no point of order p except the neutral element [4, 8]. Furthermore $g+1$ is also the number of "new" finite singularities which the invariant transformation equation of degree p acquires by reduction modulo p.

Here $\pi_2(n)$ and $\pi_3(n)$ are the numbers of incongruent solutions of $x^2+1\equiv 0$ and $x^2+x+1\equiv 0$ both modulo n. Therefore $\pi_2(n)$ is zero if n is a multiple of 4; otherwise we have

$$\pi_2(n)=\prod_{p\mid n}(1+(-4/p)).$$

Similarly $\pi_3(n)$ is zero if n is a multiple of 9; otherwise we have

$$\pi_3(n)=\prod_{p\mid n}(1+(-3/p)).$$

Since the theorem is true for $n=2$, we shall assume that n is greater than 2. If we decompose $LF(2,n)$ into left cosets by H, the coset space can be identified with "projective line" over integers modulo n. This space is a homogeneous space with $LF(2,n)$ as structure group and with H as isotropy group. Suppose that T is an inertia group of $k(j,Ku(\Omega))$ over $k(j)$ at $j=a$. Then T operates on this homogeneous space, hence the space splits into several orbits such that, of course, each orbit is a homogeneous space with T as structure group. We call an orbit to be exceptional if the order of the corresponding isotropy group, which we call the multiplicity of the orbit, is greater than 1. According to Hilbert's Galois theory, there is a one-to-one correspondence between orbits and prime divisors of Σ lying above $j=a$. Moreover a prime divisor of Σ will be ramified in $k(j,Ku(\Omega))$ with index e if and only if the corresponding orbit has multiplicity e. Now in case $a=\infty$ the cyclic group of order n generated by $\pm(1\ 1,0\ 1)$ can be taken as T. Then, if we decompose n in the form mm', a point of our "projective line" having m as its second co-ordinate belongs to an orbit of multiplicity $m(m,m')$. Moreover the number of orbits of this type is $\phi((m,m'))$. Therefore, if $\mathfrak{d}$ is the different of $k(j,Ku(\Omega))$ over Σ, the contribution to $\deg(\mathfrak{d})$ of the prime divisors of $k(j,Ku(\Omega))$ lying above $j=\infty$ is given by

$$\sum_{mm'=n}\phi((m,m'))(1-1/m(m,m'))\cdot 1/2\cdot n\phi(n)$$
$$=(\sum_{mm'=n}\phi((m,m'))-\psi(n)/n)\cdot 1/2\cdot n\phi(n).$$

In order to determine contributions of other prime divisors, we observe the following. Consider an inertia group T of $k(j,Ku(\Omega))$ over $k(j)$ at $j=12^3$. Then elements of order 2 in T are conjugate to $\pm(0\ -1,1\ 0)$ in $LF(2,n)$. Similarly elements of order 3 in an inertia group of $k(j,Ku(\Omega))$ over $k(j)$ at $j=0$ are conjugate to either $\pm(-1\ -1,1\ 0)$ or $\pm(0\ 1,-1\ -1)$ in $LF(2,n)$. Since the proofs are similar, we shall prove the first statement. As we can see, it is sufficient to prove the statement for the direct limit of

all $k(j, Ku(\Omega))$. This case is, then, reduced to its p-primary part whose Galois group is the inverse limit of $LF(2, p^e)$ for $e = 1, \cdots$, i.e. the group of linear fractional transformations $x \to (ax+b)(cx+d)^{-1}$. Here a, b, c, d are p-adic integers satisfying $ad - bc = 1$. An element of order 2 of this group is then of the form $x \to (ax+b)(cx-a)^{-1}$ with $a^2 + bc + 1 = 0$. However an elementary number theory shows that this element is conjugate to $x \to -1/x$ in this group. As a consequence, in the case of characteristic 3, the whole inertia group of $k(j, Ku(\Omega))$ over $k(j)$ at $j = 0$ can not be contained in H. Also, in the case of characteristic 2, the Klein four group in the inertia group of $k(j, Ku(\Omega))$ over $k(j)$ at $j = 0$ can not be contained in H. Therefore, in every case, the multiplicities of exceptional orbits above $j = 12^3$ and $j = 0$ are 2 and 3. If the characteristic is different from 2 and 3, we have $\pi_2(n)$ exceptional orbits of multiplicity 2 and $\pi_3(n)$ exceptional orbits of multiplicity 3. If the characteristic is 3, we still have $\pi_2(n)$ exceptional orbits of multiplicity 2 but $1/2 \cdot \pi_3(n)$ exceptional orbits of multiplicity 3. Similarly, if the characteristic is 2, we have $1/2 \cdot \phi_2(n)$ exceptional orbits of multiplicity 2 and $\pi_3(n)$ exceptional orbits of multiplicity 3. However, just because of the wildness of ramification in the second and the third cases, the contributions to $\deg(\mathfrak{d})$ are the same in these three cases, and we get $(1/2 \cdot \pi_2(n) + 2/3 \cdot \pi_3(n)) \cdot 1/2 \cdot n\phi(n)$. If we sum up the contributions to $\deg(\mathfrak{d})$ in the relative genus formula

$$1/12 \cdot (n-6)\phi(n)\psi(n) = \deg(\mathfrak{d}) + 1/2 \cdot n\phi(n)(2g-2),$$

we get the genus formula stated in the theorem.

In the course of the above proof we considered the direct limit of fields of modular functions of levels $p, p^2, \cdots$ and its Galois group over $k(j)$ with Krull's topology, which is the group of linear fractional transformations

$$x \to (ax+b)(cx+d)^{-1}$$

with p-adic integer coefficients a, b, c, d satisfying $ad - bc = 1$. This we call the p-primary part of the *abstract modular group*.

Finally we shall explain the meaning of the field Σ. Suppose that ω is a primitive n-th division point of A_j. Here n is not a multiple of the characteristic. Then ω generates a cyclic subgroup $g(\omega)$, say, of A_j of order n. This cyclic group contains $\phi(n)$ primitive n-th division points by which we can construct a cycle $c(\omega)$. We note that $g(\omega)$ and $c(\omega)$ determine each other uniquely. The set of all primitive n-th division points of A_j is divided into $\psi(n)$ cycles like $c(\omega)$ with disjoint supports. If Ω is, as before, the group of points of A_j of order n, then $k(j, c(\omega))$ is a subfield of $k(j, Ku(\Omega))$.

Moreover the subgroup of $LF(2,n)$ which corresponds to this subfield is contained in H. Since $c(\omega)$ has at most $\psi(n)$ conjugates over $F(j)$, hence over $k(j)$, we see that it has exactly $\psi(n)$ conjugates over $k(j)$ and that the Galois group of $k(j, Ku(\Omega))$ over $k(j, c(\omega))$ is H. In other words we have $\Sigma = k(j, c(\omega))$. On the other hand the factor group $A_j/g(\omega)$ is an elliptic curve defined over $F(j, c(\omega))$. Therefore the corresponding absolute invariant j_ω is contained in $F(j, c(\omega))$. However, since $c(\omega)$ is separable over $F(j, j_\omega)$ and has no other conjugate than itself over $F(j, j_\omega)$, we see that it is rational over this field. In other words we have $F(j, j_\omega) = F(j, c(\omega))$. Moreover, since we have

$$[F(j, j_\omega) : F(j)] = [k(j, j_\omega) : k(j)] = \psi(n),$$

we see that $F(j, j_\omega)$ is a regular extension of F. Therefore, if $\Phi(j_\omega, j) = 0$ is an irreducible equation over F, then $\Phi_n(X, Y)$ is *absolutely irreducible.* The equation $\Phi_n(X, Y) = 0$ is what we call the *invariant transformation equation* of degree n. If we adjoin one root of $\Phi_n(X, j) = 0$ to $k(j)$, we get the field Σ. If we adjoin all roots to $k(j)$, we get a subfield of $k(j, Ku(\Omega))$. The above consideration shows that the subgroup of $LF(2,n)$ which corresponds to this subfield is the group of scalar matrices in $SL(2,n)$ modulo Z. Therefore, over this splitting field of $\Phi_n(X, j) = 0$ the field of modular functions of level n is composed of quadratic extensions the number of which is the half of the number of incongruent solutions of $x^2 \equiv 1 \bmod n$.

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A THEOREM OF COMPLETENESS OF CHARACTERISTIC SYSTEMS OF COMPLETE CONTINUOUS SYSTEMS.*

By K. Kodaira and D. C. Spencer.[1]

Dedicated to Professor Oscar Zariski on the occasion of his sixtieth birthday.

1. Introduction. The theorem of completeness of characteristic systems of complete continuous systems of curves on algebraic surfaces, conceived by Italian geometers around 1900 (see [8], no. 4, pp. 39-42), was first proved in 1921 by Severi [6] under the assumption that the curves are arithmetically effective. His proof is based on a theorem of fundamental importance due to Poincaré [9], [10] (see also Zariski [12]). Later, Severi [7] removed the assumption of arithmetical effectiveness and proved the theorem of completeness for semi-regular curves. We remark that a curve C on a surface is called semi-regular by Severi if the canonical linear system of the surface cuts out on C a complete linear system. In Section 2, we formulate this concept of semi-regularity in terms of sheaves in a form which is applicable to submanifolds of co-dimension 1 of higher dimensional complex manifolds.

A theorem of completeness of characteristic systems of complete continuous systems of submanifolds of co-dimension 1 on higher dimensional algebraic manifolds was proved by Kodaira [2], [3], under the assumption that the systems are "ample" in a certain sense. However, his proof, based essentially on the theory of harmonic differential forms, is indirect and does not reveal the real nature of the theorem. Moreover, his theorem does not cover Severi's result mentioned above since his "ampleness" is stronger than Severi's semi-regularity.

The purpose of this note is to prove, by an elementary direct method, the theorem of completeness of characteristic systems of complete continuous

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systems for semi-regular submanifolds of co-dimension 1 of higher dimensional manifolds. Our result covers as special cases both the theorem of Severi and that of Kodaira, and it holds also for an arbitrary non-algebraic complex manifold which need not even be compact. Moreover, our direct method of proof based on the theory of sheaves is quite elementary and involves only local considerations confined to a neighborhood of a given submanifold. The argument shows clearly that semi-regularity is just the right condition for the theorem of completeness. We remark finally that it seems impossible to find any *useful* necessary and sufficient condition for the theorem of completeness.

2. Formulation of the theorem. By a complex analytic fibre space we mean a triple $(\mathcal{V}, \varpi, M)$ of connected complex manifolds $\mathcal{V}$, M and a holomorphic map ϖ of $\mathcal{V}$ onto M. A fibre $V_t = \varpi^{-1}(t)$, $t \in M$, of the complex analytic fibre space will be called singular if there exists a point $p \in V_t$ such that the rank of the Jacobian of the map ϖ at p is less than the (complex) dimension of M. A complex analytic family of compact, complex manifolds is, by definition, a complex analytic fibre space without singular fibres whose fibres are compact and connected (see Kodaira and Spencer [4], §1). Now let W be a paracompact (but not necessarily compact) complex manifold. *By a complex analytic family of compact submanifolds of W we mean a complex analytic family $\mathcal{V} \xrightarrow{\varpi} M$ of compact complex manifolds such that each fibre $V_t = \varpi^{-1}(t)$ is a complex submanifold of W together with a holomorphic map Φ of $\mathcal{V}$ into W whose restriction to each fibre V_t is the inclusion map $V_t \to W$* (see Kodaira and Spencer [4], §12).[2] Denote a point on $\mathcal{V}$ by p. Clearly $p \to (\Phi(p), \varpi(p))$ is a biregular map of $\mathcal{V}$ into $W \times M$. We identify p with $(\Phi(p), \varpi(p)) \in W \times M$ and consider $\mathcal{V}$ as a *submanifold* of $W \times M$. Let $\pi\colon W \times M \to M$ be the canonical projection of $W \times M$ onto M. Then ϖ coincides with the restriction of π to $\mathcal{V}$: $\varpi = \pi \mid \mathcal{V}$. Conversely, *we may define a complex analytic family of compact submanifolds of W as a submanifold $\mathcal{V}$ of $W \times M$ such that $(\mathcal{V}, \pi \mid \mathcal{V}, M)$ forms a complex analytic family of compact complex manifolds* (see Weil [11], p. 881). In fact, the canonical projection $W \times M \to W$ induces a holomorphic map Φ of $\mathcal{V}$ into W, whose restriction to each fibre $V_t = \mathcal{V} \cap \pi^{-1}(t)$ is an inclusion map $V_t \to W$. In this note, we are exclusively concerned with complex analytic families of submanifolds of W of co-dimension 1.

Let $\mathcal{V} \xrightarrow{\varpi} M$ be a complex analytic family of compact submanifolds

[2] It is assumed in [4] that W is compact, but compactness of W is not required in the argument used here.

of a paracompact complex manifold W of co-dimension 1. We denote the (complex) dimension of the fibres $V_t = \varpi^{-1}(t)$ of $\mathfrak{V}$ by n; the dimension of W is therefore $n+1$. We may suppose that $\mathfrak{V} \subset W \times M$ and that $\varpi = \pi \mid \mathfrak{V}$, as was mentioned above. Clearly, $\mathfrak{V}$ is a submanifold of $W \times M$ of co-dimension 1, i.e., a *divisor* on $W \times M$. The divisor $\mathfrak{V}$ determines a complex line bundle $F = [\mathfrak{V}]$ over $W \times M$ (see for example Kodaira [3], p. 718). The restriction $F_t = F \mid W \times t$ of F to $\pi^{-1}(t) = W \times t$ may be considered as a complex line bundle over W by an obvious identification of $W \times t$ with W. We have $F_t = [V_t]$, since $V_t \times t$ is the divisor on $W \times t$ cut out by $\mathfrak{V}$. We denote by $\Omega(F_t)$ the sheaf over W of holomorphic sections of F_t and by Ψ_t the restriction of $\Omega(F_t)$ to V_t.

We denote by w a point on W and by $(w^1, w^2, \cdots, w^{n+1})$ the local holomorphic coordinates (not specified) of w. Consider a small "spherical" neighborhood N of a point on M and let $\{U_i\}$ be a locally finite covering of W by sufficiently small coordinate neighborhoods U_i. The submanifold $\mathfrak{V}$ of $W \times M$ is defined in each neighborhood $U_i \times N$ by a holomorphic equation

$$S_i(w,t) = 0, \tag{1}$$

where $S_i(w,t)$ is a holomorphic function on $U_i \times N$ such that

$$\sum_{\alpha=1}^{n+1} | \partial S_i(w,t)/\partial w^\alpha |^2 \neq 0$$

at each point (w,t) of $\mathfrak{V} \cap U_i \times N$ (if $\mathfrak{V} \cap U_i \times N$ is empty, we set $S_i(w,t) = 1$). Letting

$$S_i(w,t) = f_{ik}(w,t) S_k(w,t), \qquad w \in U_i \cap U_k, \tag{2}$$

we obtain a system $\{f_{ik}(w,t)\}$ of non-vanishing holomorphic functions $f_{ik}(w,t)$ defined, respectively, on $U_i \times N \cap U_k \times N$ satisfying

$$f_{ik}(w,t) = f_{ij}(w,t) f_{jk}(w,t), \qquad \text{for } w \in U_i \cap U_j \cap U_k. \tag{3}$$

Clearly, $\{f_{ik}(w,t)\}$ defines the complex line bundle $F \mid W \times N$ (the restriction of F to $W \times N$). Moreover, for each point $t \in N$, the submanifold V_t of W is defined in U_i by the holomorphic equation $S_i(w,t) = 0$ and the complex line bundle $F_t = [V_t]$ is defined by $\{f_{ik}(w,t)\}$.

Denote by $(t_1, \cdots, t_r, \cdots, t_m)$ a system of holomorphic coordinates on N. For any tangent vector

$$v = \sum_{r=1}^{m} v_r (\partial/\partial t_r)$$

of M at $t \in N$, we set

(4) $$\psi_i(w,v) = -vS_i(w,t), \qquad \text{for } w \in V_t \cap U_i,$$
where
$$vS_i(w,t) = \sum_{r=1}^{m} v_r \partial S_i(w,t)/\partial t_r.$$
Since $S_i(w,t) = 0$ for $w \in V_t$, it follows from (2) that

(5) $$\psi_i(w,v) = f_{ik}(w,t)\psi_k(w,v) \qquad \text{for } w \in V_t \cap U_i \cap U_k.$$

This shows that
$$\psi(v) : w \to (w, \psi_i(w,v))$$
is a holomorphic section of F_t over V_t. Thus, for each tangent vector v of M at t, we obtain an element $\psi(v)$ of $H^0(V_t, \Psi_t)$. We call $\psi(v) \in H^0(V_t, \Psi_t)$ *the infinitesimal displacement of* V along v and denote it by $\rho_{d,t}(v)$ (see Kodaira and Spencer [4], § 12). Denote by T_t the tangent space of M at t. It is clear that

(6) $$\rho_{d,t} : v \to \rho_{d,t}(v) = \psi(v)$$

is a linear map of T_t into $H^0(V_t, \Psi_t)$.

For any element ψ of $H^0(V_t, \Psi_t)$ we denote by (ψ) the divisor of ψ. The set of all divisors (ψ), $\psi \in H^0(V_t, \Psi_t)$, forms a complete linear system L_t on V_t (see Kodaira [3], p. 718). The *characteristic system* of the family $\mathcal{V} \xrightarrow{\varpi} M$ on V_t is, by definition, the linear subsystem of L_t consisting of the divisors $(\rho_{d,t}(v))$ of infinitesimal displacements $\rho_{d,t}(v)$, $v \in T_t$ (see Kodaira [3], p. 738), and is said to be *complete* if and only if it coincides with the complete linear system L_t. In view of the one-to-one correspondonce between linear subsystems of L_t and linear subspaces of $H^0(V_t, \Psi_t)$, *the characteristic system of* $\mathcal{V} \xrightarrow{\varpi} M$ *on* V_t *is complete if and only if the map* $\rho_{d,t} : T_t \to H^1(V_t, \Psi_t)$ *is surjective.*

Let $\mathcal{V} \xrightarrow{\varpi} M$ be a complex analytic family of compact submanifolds of W and let t_0 be a point on M. We say that $\mathcal{V} \xrightarrow{\varpi} M$ is *maximal* at t_0 if, for any complex analytic family $\mathcal{V}' \xrightarrow{\varpi'} M'$ of compact submanifolds of W such that $\varpi^{-1}(t_0) = \varpi'^{-1}(t'_0)$, $t'_0 \in M'$, there exists a holomorphic map h of a neighborhood N' of t'_0 on M' into M which maps t'_0 into t_0 such that $\varpi'^{-1}(t') = \varpi^{-1}(h(t'))$ for $t' \in N'$, where we indicate by $\varpi'^{-1}(t') = \varpi^{-1}(t)$ that $\varpi'^{-1}(t')$ and $\varpi^{-1}(t)$ are the *same* submanifold of W. We note that, if $\varpi'^{-1}(t') = \varpi^{-1}(h(t'))$ for $t' \in N'$, there exists a holomorphic map h_* of $\mathcal{V}' \mid N' = \varpi'^{-1}(N')$ into $\mathcal{V}$ which maps each fibre $\varpi'^{-1}(t')$ biregularly onto $\varpi^{-1}(h(t'))$ and therefore $\mathcal{V}' \mid N'$ is the family induced from $\mathcal{V}$ by the map h. In fact, if we denote by $\hat{h}$ the holomorphic map: $(w, t') \to (w, h(t'))$ of

$W \times N'$ into $W \times M$ and if we suppose that $\mathfrak{V}' \mid N' \subset W \times N'$, $\mathfrak{V} \subset W \times M$, the restriction of $\hat{h}$ to $\mathfrak{V}' \mid N'$ gives h_*. Thus, if $\mathfrak{V} \xrightarrow{\varpi} M$ is maximal at t_0, $\mathfrak{V} \xrightarrow{\varpi} M$ is complete relative to W at t_0 (see Kodaira and Spencer [4], § 12).

Now we formulate our main theorem. In what follows we denote by M a spherical domain:

$$M = \{t \mid t = (t_1, t_2, \cdots, t_m), \qquad |t_1|^2 + \cdots + |t^2_m| < \delta\},$$

where $\delta > 0$. Let V_0 be a compact submanifold of W of co-dimension 1 and let $F_0 = [V_0]$ be the complex line bundle over W determined by V_0. Moreover, let $\Omega(F_0)$ be the sheaf over W of holomorphic sections of F_0 and let Ψ_0 be the restriction of $\Omega(F_0)$ to V_0. We denote by r_0 the restriction map: $\Omega(F_0) \to \Psi_0$. r_0 induces a homomorphism $r_0^*: H^1(W, \Omega(F_0)) \to H^1(V_0, \Psi_0)$ in a canonical manner.

DEFINITION. *We say that V_0 is semi-regular if the image*

$$r_0^* H^1(W, \Omega(F_0))$$

is zero.

In case W is an algebraic surface, our definition of semi-regularity is equivalent to Severi's definition mentioned in Section 1. To show this, let K be the canonical bundle on W and let $K_0 = K \mid V_0$ be the restriction of K to V_0. We remark that the canonical system $|K|$ cuts out on V_0 a complete linear system if and only if the restriction map

$$r_0: H^0(W, \Omega(K)) \to H^0(V_0, \Omega(K_0))$$

is surjective. Consider the exact sequence

$$0 \to \Omega \to \Omega(F_0) \xrightarrow{r_0} \Psi_0 \to 0,$$

where Ω denotes the sheaf over W of germs of holomorphic functions. By the duality theorem, the corresponding exact cohomology sequence

$$\cdots \to H^1(W, \Omega(F_0)) \xrightarrow{r_0^*} H^1(V_0, \Psi_0) \to H^2(W, \Omega) \to \cdots$$

is *dual* to

$$\cdots \leftarrow H^1(W, \Omega(K \otimes F_0^{-1})) \leftarrow H^0(V_0, \Omega(K_0)) \xleftarrow{r_0} H^0(W, \Omega(K)) \leftarrow \cdots.$$

It follows that the image $r_0^* H^1(W, \Omega(F_0))$ is zero if and only if r_0: $H^0(W, \Omega(K)) \to H^0(V_0, \Omega(K_0))$ is surjective.

MAIN THEOREM. *If a compact submanifold V_0 of W of codimension* 1 *is semi-regular, then there exists a complex analytic family $\mathcal{V} \xrightarrow{\varpi} M$ of compact submanifolds of W containing V_0 as the fibre $\varpi^{-1}(0)$ over $0 \in M$ such that, for each point $t \in M$, $\rho_{d,t}$ maps the tangent space T_t of M at t isomorphically onto $H^0(V_t, \Psi_t)$, where $V_t = \varpi^{-1}(t)$. Moreover, the family $\mathcal{V} \xrightarrow{\varpi} M$ is maximal at each point t of M.*

Now we apply our main theorem to the case in which W is an algebraic manifold imbedded in a projective space. Let V_0 be a submanifold of W of codimension 1. It is well known that the set $\mathcal{S}$ of all effective divisors $X \approx V_0$ on W, where $\approx$ denotes homology with integer coefficients, forms an algebraic system, i. e., there exist a (possibly reducible and singular) algebraic variety Σ and a one-to-one algebraic correspondence $\sigma \to X = X_\sigma$ between Σ and $\mathcal{S}$ (see Weil [11], p. 887). Σ is called the parameter variety of $\mathcal{S}$. The parameter variety Σ may be determined in the following manner (see Weil [11], p. 887; cf. also Kodaira [3], pp. 734-735). Take a hypersurface section E of W of sufficiently high order and consider the algebraic system $\mathcal{B}$ of all effective divisors $D \approx V_0 + E$ on W. The parameter variety Λ of $\mathcal{B}$ is a non-singular algebraic variety imbedded in a projective space. Let $\lambda \to D_\lambda$ be the one-to-one algebraic corerspondence between Λ and $\mathcal{B}$. Moreover, denote by $D_\lambda > E$ that $D_\lambda - E$ is effective. Then Σ is the subvariety of Λ consisting of all points λ such that $D_\lambda > E$ and, for each point $\sigma \in \Sigma \subset \Lambda$, the corresponding divisor $X_\sigma \in \mathcal{S}$ is given by $X_\sigma = D_\sigma - E$. We call the parameter variety Σ thus determined the *canonical parameter variety* of $\mathcal{S}$.

Suppose that V_0 is semi-regular and let $\mathcal{V} \xrightarrow{\varpi} M$ be the complex analytic family given in our main theorem. It follows from the bijectivity of $\rho_{d,t}$ that $V_t \neq V_s$ for $t \neq s$, provided that t and s are contained in a small neighborhood of a point on M. Hence, replacing M by a smaller spherical subdomain, if necessary, we may assume that $V_t \neq V_s$ for $t \neq s$, $t \in M$, $s \in M$. Since $V_t \approx V_0$, there exists for each $t \in M$ a point $\lambda(t) \in \Sigma \subset \Lambda$ such that $V_t = X_{\lambda(t)}$. By a result of Weil [11], the map $t \to \lambda(t)$ thus defined is *analytic* in the sense that the graph of the map $t \to \lambda(t)$ is an analytic subvariety of $M \times \Lambda$, while $t \to \lambda(t)$ is obviously one-to-one and continuous. It follows that $t \to \lambda(t)$ is regular analytic. Moreover, we infer from the bijectivity of $\rho_{d,t}$ that $t \to \lambda(t)$ *is biregular.* In fact, let N be a small neighborhood of a point t_0 on M and let $\{U_i\}$ be a finite covering of W. Moreover, let $S_i(w, t)$ be the holomorphic functions which defines $\mathcal{V}$ on $U_i \times N$ (see (1)). Let B be a neighborhood of $\lambda(t_0)$ on Λ. For $\lambda \in B$ the divisor D_λ is defined in each U_i by

$$R_i(w, \lambda) = 0,$$

where $R_i(w, \lambda)$ is a holomorphic function of w and λ. It follows from $D_{\lambda(t)} = V_t + E$ that

$$f_i(w, t) = R_i(w, \lambda(t))/S_i(w, t)$$

is a holomorphic function in w and t and that $f_i(w, t)$ does not vanish if $E \cap U_i$ is empty. Applying $\partial/\partial t_r$, we obtain from

$$f_i(w, t) S_i(w, t) = R_i(w, \lambda(t))$$

the equality

$$f_i(w, t)\psi_i(w, \partial/\partial t_r) = -\sum_\nu (\partial\lambda_\nu(t)/\partial t_r) \cdot (\partial R_i(w, \lambda(t))/\partial\lambda_\nu) \quad \text{for } w \in V_t \cap U_i, \tag{7}$$

where $(\lambda_1, \cdots, \lambda_\nu, \cdots)$ denotes the local coordinates of λ. The bijectivity of $\rho_{d,t}$ implies that $\psi(\partial/\partial t_r) = \rho_{d,t}(\partial/\partial t_r)$, $r = 1, 2, \cdots, m$, are linearly independent. Therefore, considering U_i such that $E \cap U_i$ is empty, we infer from (7) that the rank of the Jacobian

$$\partial(\lambda_1, \lambda_2, \cdots, \lambda_\nu, \cdots)/\partial(t_1, t_2, \cdots, t_m)$$

of the map $t \to \lambda(t)$ is equal to m. This proves that $t \to \lambda(t)$ is biregular.

Now, since $\mathfrak{V} \xrightarrow{\varpi} M$ is maximal at each point $t \in M$, we infer that the image $\lambda(M) \subset \Sigma$ is an *open* subset of Σ. Thus the point $\lambda(0)$ has the neighborhood $\lambda(M)$ on Σ which is analytically homeomorphic to the spherical domain M and hence $\lambda(0)$ is a simple point of Σ. Moreover the restriction $\mathcal{S} \mid \lambda(M) = \{X_\sigma \mid \sigma \in \lambda(M)\}$ of the algebraic system $\mathcal{S}$ to $\lambda(M)$ is a complex analytic family which is complex analytically eqivalent to $\mathfrak{V} \to M$. Hence the characteristic system of $\mathcal{S}$ on $X_{\lambda(0)} = V_0$ can be defined and coincides with the characteristic system of $\mathfrak{V}$ on V_0. It is therefore complete.

A *continuous system* is, by definition, an algebraic system whose parameter variety is irreducible. A continuous system is called *complete* if it is maximal in the sense that it is not contained in a larger continuous system. The parameter variety $\Sigma \subset \Lambda$ is composed of a finite number of irreducible components:

$$\Sigma = \Sigma' \cup \Sigma'' \cup \cdots \cup \Sigma^{(l)} \cup \cdots.$$

Corresponding to this, $\mathcal{S}$ is composed of a finite number of complete continuous systems $\mathcal{S}', \mathcal{S}'', \cdots, \mathcal{S}^{(l)}, \cdots$, where $\mathcal{S}^{(l)} = \mathcal{S} \mid \Sigma^{(l)} = \{X_\sigma \mid \sigma \in \Sigma^{(l)}\}$. It is clear that the simple point $\lambda(0)$ of Σ belongs to one and only one irreducible component, say Σ'. Hence $V_0 = X_{\lambda(0)}$ belongs to one and only one complete continuous system $\mathcal{S}'$. Thus we obtain

THEOREM (*Completeness of characteristic systems*). *Let W be an algebraic manifold imbedded in a projective space and let V_0 be a submanifold of W of codimension 1. If V_0 is semi-regular, then V_0 belongs to only one complete continuous system $\mathcal{S}'$ of effective divisors on W and V_0 corresponds to a simple point of the canonical parameter variety Σ' of $\mathcal{S}'$. Moreover, the characteristic system of $\mathcal{S}'$ on V_0 is complete.*

This theorem covers as special cases both the theorem of Severi [7] and that of Kodaira [3].

3. Construction of a complex analytic family. Let W be a paracompact complex manifold of complex dimension $n+1\geqq 2$ and let V_0 be a compact submanifold of W of dimension n. In what follows we denote by p a point on W and by $(w^1(p), w^2(p),\cdots, w^{n+1}(p))$ the coordinate of p with respect to a system of local holomorphic coordinates $(w^1, w^2,\cdots, w^{n+1})$ on W. We choose a locally finite covering $\mathfrak{U}=\{U_i\}$ of W such that i) each neighborhood U_i is a polycylinder:

$$U_i=\{p\mid |w^1_i(p)|<1, |w^2_i(p)|<1,\cdots, |w^{n+1}_i(p)|<1\},$$

where $(w^1_i, w^2_i,\cdots, w^{n+1}_i)$ is a system of local holomorphic coordinates which covers the closure of U_i (thus the closure of U_i is compact), ii) $V_0\cap U_i$ coincides with the coordinate plane $w^{n+1}_i=0$ if $V_0\cap U_i$ is not empty, and such that iii) $V_0\cap U_i\cap U_k$ is not empty if $V_0\cap U_i$, $V_0\cap U_k$ and $U_i\cap U_k$ are not empty. Moreover, if $V_0\cap U_i$ is not empty, we write for convenience

$$w_i=w^{n+1}_i, z^1_i=w^1_i,\cdots, z^n_i=w^n_i.$$

Let

$$S_{i|0}(p)=\begin{cases} w_i(p), & \text{if } V_0\cap U_i\neq\text{empty},\\ 1\ , & \text{if } V_0\cap U_i=\text{empty},\end{cases}$$

and let

$$f_{ik|0}(p)=S_{i|0}(p)/S_{k|0}(p), \qquad \text{for } p\in U_i\cap U_k. \tag{8}$$

Clearly, $f_{ik|0}(p)$ are non-vanishing holomorphic functions defined on $U_i\cap U_k$, respectively, and the system $\{f_{ik|0}(p)\}$ defines the complex line bundle $F_0=[V_0]$. More precisely, we have the product representation

$$F_0\mid U_i=U_i\times \boldsymbol{C} \tag{9}$$

of each piece $F_0\mid U_i$, where $(p,\zeta_i)\in U_i\times\boldsymbol{C}$ is identical with $(p,\zeta_k)\in U_k\times\boldsymbol{C}$ if and only if $\zeta_i=f_{ik|0}(p)\zeta_k$. Let Ψ_0 be the restriction of $\Omega(F_0)$ to V_0 and let $\{\beta_1,\cdots,\beta_r,\cdots,\beta_m\}$ be a base of the linear space $H^0(V_0,\Psi_0)$. Each

β_r is a holomorphic section of F_0 over V_0 and therefore β_r is written in the form

$$\beta_r\colon p \to (p, \beta_{ri}(p))$$

with respect to the product representation (9), where $\beta_{ri}(p)$ are holomorphic functions on $V_0 \cap U_i$ satisfying

$$\beta_{ri}(p) = f_{ik|0}(p)\beta_{rk}(p), \qquad \text{for } p \in V_0 \cap U_i \cap U_k. \tag{10}$$

First we prove the following

THEOREM 1. *If V_0 is semi-regular, then there exists a complex analytic family $\mathfrak{V} \xrightarrow{\varpi} M$ of compact submanifolds of W containing V_0 as the fibre $\varpi^{-1}(0)$ over $0 \in M$ such that $\rho_{d,0}$ maps the tangent space T_0 of M at 0 isomorphically onto $H^0(V_0, \Psi_0)$.*

Let N be a spherical neighborhood of 0 on the space of m complex variables $t_1, t_2, \cdots, t_m$, where $m = \dim H^0(V_0, \Psi_0)$. In order to prove the above Theorem 1, is suffices to construct a system $\{S_i(p,t)\}$ of holomorphic functions $S_i(p,t)$ defined respectively on $U_i \times N$ and a system $\{f_{ik}(p,t)\}$ of non-vanishing holomorphic functions $f_{ik}(p,t)$ defined respectively on $U_i \cap U_k \times N$ such that

$$S_i(p,t) = f_{ik}(p,t) S_k(p,t), \qquad \text{for } p \in U_i \cap U_k, \tag{11}$$

$$S_i(p,0) = S_{i|0}(p), \qquad f_{ik}(p,0) = f_{ik|0}(p), \tag{12}$$

$$S_i(p,t) \neq 0, \qquad \text{if } V_0 \cap U_i = \text{empty}, \tag{12a}$$

$$\partial S_i(p,t)/\partial t_r \,|_{t=0} = \beta_{ri}(p). \tag{13}$$

In fact, letting $M \subset N$ be a sufficiently small spherical subdomain with the center 0, $\mathfrak{V} \to M$ is given as the submanifold $\mathfrak{V} \subset W \times M$ defined by the holomorphic equations $S_i(p,t) = 0$.

We write $S_i(p,t)$, $f_{ik}(p,t)$ in the forms

$$S_i(p,t) = S_{i|0}(p) + \sum_{\mu=1}^{\infty} S_{i|\mu}(p,t),$$

$$f_{ik}(p,t) = f_{ik|0}(p) + \sum_{\mu=1}^{\infty} f_{ik|\mu}(p,t),$$

where $S_{i|\mu}(p,t)$, $f_{ik|\mu}(p,t)$ are homogeneous polynomials in $t = (t_1, t_2, \cdots, t_m)$ of degree μ whose coefficients are holomorphic functions on U_i, $U_i \cap U_k$, respectively. Let

$$S^{\mu}{}_i(p,t) = S_{i|0}(p) + \sum_{\lambda=1}^{\mu} S_{i|\lambda}(p,t), \tag{14}$$

$$f^{\mu}{}_{ik}(p,t) = f_{ik|0}(p) + \sum_{\lambda=1}^{\mu} f_{ik|\lambda}(p,t). \tag{15}$$

Moreover, for arbitrary power series $P(t)$, $Q(t)$, in t, we indicate by $P(t) \underset{\mu}{\equiv} Q(t)$ that $P(t) - Q(t)$ contains no terms of degree $\leqq \mu$ in t. Clearly, the equality (11) is equivalent to the system of congruences

$$S^{\mu}{}_i(p,t) \underset{\mu}{\equiv} f^{\mu}{}_{ik}(p,t) S^{\mu}{}_k(p,t), \qquad \mu = 1,2,3,\cdots. \tag{16$_\mu$}$$

In order to construct $S_{i|\mu}(p,t)$, $f_{ik|\mu}(p,t)$ by induction on μ, we assume the following special forms for $S_{i|\mu}(p,t)$, $\mu \geqq 1$:

$$S_{i|\mu}(p,t) = \begin{cases} \psi_{i|\mu}(z_i(p),t), & \text{if } V_0 \cap U_i \neq \text{empty}, \\ 0 \quad , & \text{if } V_0 \cap U_i = \text{empty}, \end{cases} \tag{17}$$

where $z_i(p) = (z^1{}_i(p), \cdots, z^n{}_i(p))$ and $\psi_{i|\mu}(z_i,t)$ is a homogeneous polynomial of degree μ in t whose coefficients are holomorphic functions of $z_i = (z^1{}_i, \cdots, z^n{}_i)$ defined on the polycylinder: $|z^1{}_i| < 1, \cdots, |z^n{}_i| < 1$.

First, we define $\psi_{i|1}(z_i,t)$ by

$$\psi_{i|1}(z_i(p),t) = \sum_{r=1}^{m} t_r \beta_{ri}(p), \qquad \text{for } p \in V_0 \cap U_i, \tag{18}$$

and determine $S_{i|1}(p,t)$, $S^1{}_i(p,t)$ by (17), (14). The congruence $(16)_1$ is equivalent to

$$S_{i|1}(p,t) = f_{ik|0}(p) S_{k|1}(p,t) + f_{ik|1}(p,t) S_{k|0}(p).$$

Hence, letting

$$f_{ik|1}(p,t) = \{S_{i|1}(p,t) - f_{ik|0}(p) S_{k|1}(p,t)\}/S_{k|0}(p),$$

we obtain $f^1{}_{ik}(p,t) = f_{ik|0}(p) + f_{ik|1}(p,t)$ satisfying $(16)_1$, provided that $f_{ik|1}(p,t)$ is holomorphic in p. Now it follows from (10) and (18) that

$$S_{i|1}(p,t) - f_{ik|0}(p) S_{k|1}(p,t) = 0 \qquad \text{for } p \in V_0 \cap U_i \cap U_k.$$

We infer therefore that $f_{ik|1}(p,t)$ is holomorphic in p.

Now suppose that $S^{\mu}{}_i(p,t)$, $f^{\mu}{}_{ik}(p,t)$ satisfying $(16)_\mu$ are already determined. Clearly, $(16)_\mu$ implies that

$$f^{\mu}{}_{ik}(p,t) \underset{\mu}{\equiv} f^{\mu}{}_{ij}(p,t) f^{\mu}{}_{jk}(p,t), \qquad p \in U_i \cap U_j \cap U_k. \tag{19$_\mu$}$$

We define homogeneous polynomials $\psi_{ik|\mu+1}(p,t)$ in t of degree $\mu+1$ whose coefficients are holomorphic functions on $V_0 \cap U_i \cap U_k$ by

$$\psi_{ik|\mu+1}(p,t) \underset{\mu+1}{\equiv} f^{\mu}{}_{ik}(p,t) S^{\mu}{}_k(p,t) - S^{\mu}{}_i(p,t) \qquad \text{for } p \in V_0 \cap U_i \cap U_k.$$

Then we have

$$(20)\qquad \psi_{ik|\mu+1}(p,t)=\psi_{ij|\mu+1}(p,t)+f_{ij|0}(p)\psi_{jk|\mu+1}(p,t), \qquad \text{for } p\in V_0\cap U_i\cap U_j\cap U_k.$$

In fact, letting $\psi_{ik|\mu+1}=\psi_{ik|\mu+1}(p,t)$, $f^{\mu}{}_{ik}=f^{\mu}{}_{ik}(p,t)$, $S^{\mu}{}_i=S^{\mu}{}_i(p,t),\cdots$, we have

$$\begin{aligned}\psi_{ij|\mu+1}+f_{ij|0}\psi_{jk|\mu+1}&\underset{\mu+1}{\equiv}f^{\mu}{}_{ij}S^{\mu}{}_j-S^{\mu}{}_i+f_{ij|0}f^{\mu}{}_{jk}S^{\mu}{}_k-f_{ij|0}S^{\mu}{}_j\\&\underset{\mu+1}{\equiv}(f^{\mu}{}_{ij}-f_{ij|0})S^{\mu}{}_j+f_{ij|0}f^{\mu}{}_{jk}S^{\mu}{}_k-S^{\mu}{}_i\\&\underset{\mu+1}{\equiv}(f^{\mu}{}_{ij}-f_{ij|0})f^{\mu}{}_{jk}S^{\mu}{}_k+f_{ij|0}f^{\mu}{}_{jk}S^{\mu}{}_k-S^{\mu}{}_i\\&\underset{\mu+1}{\equiv}f^{\mu}{}_{ij}f^{\mu}{}_{jk}S^{\mu}{}_k-S^{\mu}{}_i,\end{aligned}$$

while

$$S^{\mu}{}_k(p,t)=\sum_{\lambda=1}^{\mu}\psi_{k|\lambda}(z_k(p),t)\underset{0}{\equiv}0 \qquad \text{for } p\in V_0\cap V_k.$$

Hence by $(19)_\mu$, we obtain, for $p\in V_0\cap U_i\cap U_k\cap U_j$,

$$\psi_{ij|\mu+1}+f_{ij|0}\psi_{jk|\mu+1}\underset{\mu+1}{\equiv}f^{\mu}{}_{ik}S^{\mu}{}_k-S^{\mu}{}_i\underset{\mu+1}{\equiv}\psi_{ik|\mu+1}.$$

This proves (20). (20) shows that the system $\{\psi_{ik|\mu+1}(p,t)\}$ is a homogeneous polynomial in t of order $\mu+1$, whose coefficients are 1-cocycles on the nerve of the covering $\mathfrak{U}\mid V_0=\{V_0\cap U_i\}$ of V_0 with coefficients in the sheaf Ψ_0. Since the polycylinders $V_0\cap U_i$ are Stein manifolds, we have the *canonical isomorphism*

$$H^1(\mathfrak{U}\mid V_0,\Psi_0)\cong H^1(V_0,\Psi_0)$$

(see Cartan [1], Leray [5]). The 1-cocycle $\{\psi_{ik|\mu+1}(p,t)\}$ represents a homogeneous polynomial $\psi_{\mu+1}(t)$ in t of degree $\mu+1$ with coefficients in $H^1(V_0,\Psi_0)$. We may call $\psi_{\mu+1}(t)$ the *μ-th obstruction.*

If the obstruction $\psi_{\mu+1}(t)$ vanishes identically, we can construct $S^{\mu+1}{}_i(p,t)$, $f^{\mu+1}{}_{ik}(p,t)$ satisfying $(16)_{\mu+1}$. In fact, in view of the canonical isomorphism mentioned above, the vanishing of $\psi_{\mu+1}(t)$ implies the existence of homogeneous polynomials $\phi_{i|\mu+1}(p,t)$ in t of degree $\mu+1$ whose coefficients are holomorphic functions on $V_0\cap U_i$ such that

$$(21)\qquad \psi_{ik|\mu+1}(p,t)=\phi_{i|\mu+1}(p,t)-f_{ik|0}(p)\phi_{k|\mu+1}(p,t), \qquad \text{for } p\in V_0\cap U_i\cap U_k.$$

We define $\psi_{i|\mu+1}(z_i,t)$ by

$$\psi_{i|\mu+1}(z_i(p),t)=\phi_{i|\mu+1}(p,t), \qquad \text{for } p\in U_0\cap U_i.$$

and determine $S_{i|\mu+1}(p,t)$ by (17), $S^{\mu+1}{}_i(p,t)$ by (14). Then letting

$$\Xi^{(\mu)}{}_{ik}(p,t) = S^{\mu}{}_i(p,t) - f^{\mu}{}_{ik}(p,t)S^{\mu}{}_k(p,t) + S_{i|\mu+1}(p,t) \\ - f_{ik|0}(p)S_{k|\mu+1}(p,t),$$

we define $f_{ik|\mu+1}(p,t)$ by

$$f_{ik|\mu+1}(p,t) \underset{\mu+1}{\equiv} \Xi^{(\mu)}{}_{ik}(p,t)/S_{k|0}(p)$$

and determine $f^{\mu+1}{}_{ik}(p,t)$ by (15). We infer from (21) that

$$\Xi^{(\mu)}{}_{ik}(p,t) \underset{\mu+1}{\equiv} 0 \qquad \text{for } p \in V_0 \cap U_i \cap U_k.$$

Hence $f_{ik|\mu+1}(p,t)$ are holomorphic in p. It is easy to verify that $S^{\mu+1}{}_i(p,t)$ and $f^{\mu+1}{}_{ik}(p,t)$ thus defined satisfy $(16)_{\mu+1}$.

Now we prove that *the obstruction* $\psi_{\mu+1}(t)$ *vanishes identically if* V_0 *is semi-regular.* For this purpose it suffices to construct a polynomial $\{\eta_{ik}(p,t)\}$ in t of degree $\mu+1$ whose coefficients form a 1-cocycle on the nerve of the covering $\mathfrak{U} = \{U_i\}$ of W with coefficients in $\Omega(F_0)$ such that

$$(22) \qquad \psi_{ik|\mu+1}(p,t) = \eta_{ik}(p,t) \qquad \text{for } p \in V_0 \cap U_i \cap U_k.$$

In fact, $\{\eta_{ik}(p,t)\}$ represents a polynomial $\eta(t)$ in t with coefficients in $H^1(W,\Omega(F_0))$ and (22) implies that $\psi_{\mu+1}(t) = r_0{}^*\eta(t)$. Hence we obtain $\psi_{\mu+1}(t) = 0$ if V_0 is semi-regular.

LEMMA 1. *For each positive integer* $\lambda \leqq \mu$, *there exist polynomials* $g^{\lambda}{}_{ik}(p,t)$ *in* t *of degree* λ *whose coefficients are holomorphic functions in* p *defined on* $U_i \cap U_k$ *such that*

$$(23)_\lambda \qquad g^{\lambda}{}_{ik}(p,t) = g^{\lambda}{}_{ij}(p,t) + g^{\lambda}{}_{jk}(p,t) \qquad \text{for } p \in U_i \cap U_j \cap U_k,$$

$$(24)_\lambda \qquad f_{ik|0}(p) \exp g^{\lambda}{}_{ik}(p,t) \underset{\lambda}{\equiv} f^{\lambda}{}_{ik}(p,t).$$

Proof. By induction on $\lambda \leqq \mu$. Assume the existence of $g^{\lambda-1}{}_{ik} = g^{\lambda-1}{}_{ik}(p,t)$ satisfying $(23)_{\lambda-1}$ and $(24)_{\lambda-1}$. In view of $(24)_{\lambda-1}$ we can determine homogeneous polynomials $g_{ik|\lambda} = g_{ik|\lambda}(p,t)$ in t of degree λ whose coefficients are holomorphic functions on $U_i \cap U_k$ such that

$$f_{ik|0} \cdot (1 + g_{ik|\lambda}) \exp g^{\lambda-1}{}_{ik} \underset{\lambda}{\equiv} f^{\lambda}{}_{ik}.$$

The congruence $(19)_\mu$ implies that $f^{\lambda}{}_{ik} \underset{\lambda}{\equiv} f^{\lambda}{}_{ij} f^{\lambda}{}_{jk}$ for $\lambda \leqq \mu$. Hence we obtain

$$1 + g_{ik|\lambda} \underset{\lambda}{\equiv} (1 + g_{ij|\lambda})(1 + g_{jk|\lambda})$$

or

$$g_{ik|\lambda} = g_{ij|\lambda} + g_{jk|\lambda}.$$

Now let

$$g^{\lambda}{}_{ik}(p,t)=g^{\lambda-1}{}_{ik}(p,t)+g_{ik|\lambda}(p,t).$$

Clearly, $g^{\lambda}{}_{ik}(p,t)$ satisfy $(23)_{\lambda}$ and $(24)_{\lambda}$, q.e.d.

We define polynomials $\hat{f}^{\mu+1}{}_{ik}=\hat{f}^{\mu+1}{}_{ik}(p,t)$ in t of degree $\mu+1$ by

$$\hat{f}^{\mu+1}{}_{ik}(p,t)\underset{\mu+1}{\equiv}f_{ik|0}(p)\exp g^{\mu}{}_{ik}(p,t).$$

It follows from $(23)_{\mu}$ and $(24)_{\mu}$ that

$$\hat{f}^{\mu+1}{}_{ik}\underset{\mu+1}{\equiv}\hat{f}^{\mu+1}{}_{ij}\hat{f}^{\mu+1}{}_{jk}, \tag{25}$$

$$\hat{f}^{\mu+1}{}_{ik}(p,t)\underset{\mu}{\equiv}f^{\mu}{}_{ik}(p,t). \tag{26}$$

Combining (26) with $(16)_{\mu}$, we obtain

$$S^{\mu}{}_{i}(p,t)\underset{\mu}{\equiv}\hat{f}^{\mu+1}{}_{ik}(p,t)S^{\mu}{}_{k}(p,t). \tag{27}$$

In view of (27) we can determine homogeneous polynomials $\eta_{ik}(p,t)$ in t of degree $\mu+1$ whose coefficients are holomorphic functions on $U_i\cap U_k$ by

$$\eta_{ik}(p,t)\underset{\mu+1}{\equiv}\hat{f}^{\mu+1}{}_{ik}(p,t)S^{\mu}{}_{k}(p,t)-S^{\mu}{}_{i}(p,t).$$

We have

$$\eta_{ik}(p,t)=\eta_{ij}(p,t)+f_{ij|0}(p)\eta_{jk}(p,t).$$

In fact, using (27) and (25), we obtain

$$\begin{aligned}\eta_{ij}+f_{ij|0}\eta_{jk}&\underset{\mu+1}{\equiv}\hat{f}^{\mu+1}{}_{ij}S^{\mu}{}_{j}-S^{\mu}{}_{i}+f_{ij|0}\hat{f}^{\mu+1}{}_{jk}S^{\mu}{}_{k}-f_{ij|0}S^{\mu}{}_{j}\\&\underset{\mu+1}{\equiv}(\hat{f}^{\mu+1}{}_{ij}-f_{ij|0})S^{\mu}{}_{j}+f_{ij|0}\hat{f}^{\mu+1}{}_{jk}S^{\mu}{}_{k}-S^{\mu}{}_{i}\\&\underset{\mu+1}{\equiv}(\hat{f}^{\mu+1}{}_{ij}-f_{ij|0})\hat{f}^{\mu+1}{}_{jk}S^{\mu}{}_{k}+f_{ij|0}\hat{f}^{\mu+1}{}_{jk}S^{\mu}{}_{k}-S^{\mu}{}_{i}\\&\underset{\mu+1}{\equiv}\hat{f}^{\mu+1}{}_{ij}\hat{f}^{\mu+1}{}_{jk}S^{\mu}{}_{k}-S^{\mu}{}_{i}\underset{\mu+1}{\equiv}\hat{f}^{\mu+1}{}_{ik}S^{\mu}{}_{k}-S^{\mu}{}_{i}\underset{\mu+1}{\equiv}\eta_{ik}.\end{aligned}$$

Thus $\{\eta_{ik}(p,t)\}$ is a polynomial in t whose coefficients are 1-cocycles on the nerve of $\mathfrak{U}=\{U_i\}$ with coefficients in $\Omega(F_0)$. Moreover, since $S^{\mu}{}_{k}(p,t)\underset{0}{\equiv}0$ for $p\in V_0\cap U_k$, we have

$$\eta_{ik}(p,t)\underset{\mu+1}{\equiv}f^{\mu}{}_{ik}(p,t)S^{\mu}{}_{k}(p,t)-S^{\mu}{}_{i}(p,t)\underset{\mu+1}{\equiv}\psi_{ik|\mu+1}(p,t)$$

for $p\in V_0\cap U_i\cap U_k$. This proves (22).

Thus, in case V_0 is semi-regular, we can construct $S^{\mu}{}_{i}(p,t)$ and $\mathfrak{f}^{\mu}{}_{ik}(p,t)$ satisfying $(16)_{\mu}$ by induction on μ, and therefore we obtain $S_i(p,t)$ and $f_{ik}(p,t)$ satisfying (11), (12), (12a) and (13) which are *formal power series* in t.

4. Proof of convergence. Consider a formal power series

$$f(t)=f(p,t)=\sum f_{h_1h_2\cdots h_m}(p)(t_1)^{h_1}(t_2)^{h_2}\cdots(t_m)^{h_m}$$

whose coefficients $f_{h_1h_2\cdots h_m}(p)$ are holomorphic functions in p defined on a domain and a power series

$$a(t)=\sum a_{h_1h_2\cdots h_m}(t_1)^{h_1}(t_2)^{h_2}\cdots(t_m)^{h_m}, \qquad a_{h_1h_2\cdots h_m}\geqq 0.$$

We indicate by $f(p,t)\ll a(t)$ that

$$|f_{h_1h_2\cdots h_m}(p)|<a_{h_1h_2\cdots h_m};$$

moreover, we write $f(t)\ll a(t)$ if $f(p,t)\ll a(t)$ for each point p in the domain.

Let

$$A(t)=(b/64c)\sum_{\mu=1}^{\infty}c^{\mu}(t_1+t_2+\cdots+t_m)^{\mu}/\mu^2,$$

where b and c are positive constants. By a simple computation we obtain

$$A(t)^2\ll(b/c)A(t). \tag{28}$$

In what follows we denote by $c_1, c_2, \cdots$ positive constants. Our purpose is to show that the above construction of the formal power series $S_i(p,t)$ and $f_{ik}(p,t)$ can be carried out in such a way that

$$S_i(t)-S_{i|0}\ll A(t),$$

$$f_{ik}(t)-f_{ik|0}\ll c_1A(t),$$

provided that we choose the constants b, c, c_1 properly.

We may assume that

$$|f_{ik|0}(p)|<c_2, \qquad \text{for } p\in U_i\cap U_k. \tag{29}$$

It follows from (10) that the $\beta_{ri}(p)$ are bounded and therefore

$$S^1{}_i(t)-S_{i|0}\ll A(t), \tag{30}$$

provided that b is sufficiently large. Now, supposing that

$$f^{\mu-1}{}_{ik}(t)-f_{ik|0}\ll c_1A(t), \tag{31}$$

$$S^{\mu}{}_i(t)-S_{i|0}\ll A(t), \tag{32}$$

we first prove

$$f^{\mu}{}_{ik}(t)-f_{ik|0}\ll c_1A(t) \tag{33}$$

and then we show that $\phi_{i|\mu+1}(p,t)$ in (21) can be chosen in such a way that

(34) $$S^{\mu+1}{}_i - S_{i|0} \ll A(t).$$

It follows from $(16)_\mu$ that

(35) $$f_{ik|\mu}(p,t) \underset{\mu}{\equiv} \{S^\mu{}_i(p,t) - f^{\mu-1}{}_{ik}(p,t)S^\mu{}_k(p,t)\}/S_{k|0}(p).$$

We have

$$S^\mu{}_i - f^{\mu-1}{}_{ik}S^\mu{}_k = S^\mu{}_i - S_{i|0} - (f^{\mu-1}{}_{ik} - f_{ik|0})(S^\mu{}_k - S_{k|0})$$
$$- f_{ik|0}(S^\mu{}_k - S_{k|0}) - S_{k|0}(f^{\mu-1}{}_{ik} - f_{ik|0}),$$

while $S^\mu{}_i - f^{\mu-1}{}_{ik}S^\mu{}_k \underset{\mu-1}{\equiv} 0$, and therefore the term $S_{k|0}(f^{\mu-1}{}_{ik} - f_{ik|0})$ contributes nothing to $S^\mu{}_i - f^{\mu-1}{}_{ik}S^\mu{}_k$. Hence, using (31), (32), (29) and (28), we obtain

(36) $$S^\mu{}_i(t) - f^{\mu-1}{}_{ik}(t)S^\mu{}_k(t) \ll A(t) + c_1A(t)^2 + c_2A(t) \ll c_3A(t),$$

where

$$c_3 = 1 + (bc_1/c) + c_2.$$

Consider the case in which $U_k \cap V_0$ is *not* empty. Let $\{U^*{}_i\}$ be a covering of W such that *the closure of each $U^*{}_i$ is contained in U_i.* It is clear that, *if $p \in U^*{}_i \cap U_k$ and if $|w_k(p)| \leqq \epsilon$, the disk*

$$\Delta = \{q \mid z_k(q) = z_k(p), |w_k(q)| \leqq \epsilon\}$$

is contained in $U_i \cap U_k$, provided that the constant $\epsilon > 0$ is sufficiently small (we recall that $\{U_i\}$ is a locally finite covering and that $(z_k, w_k) = (z^1{}_k, \cdots, z^n{}_k, w_k)$ is a system of holomorphic coordinates which covers U_k and U_k is the polycylinder: $|z^1{}_k| < 1, \cdots, |z^n{}_k| < 1$, $|w_k| < 1$). We infer that

(37) $$f_{ik|\mu}(p,t) \ll (c_3/\epsilon)A(t), \qquad \text{for } p \in U^*{}_i \cap U_k.$$

In fact, in case $|w_k(p)| \geqq \epsilon$, (37) follows immediately from (35) and (36), since $S_{k|0}(p) = w_k(p)$. In case $|w_k(p)| < \epsilon$, we observe that each coefficient of $f_{ik|\mu}(t)$ is holomorphic on the disk $\Delta \subset U_i \cap U_k$ and that the estimate (37) is valid for each point on the periphery of Δ. Hence, by the maximum principle, (37) is valid for $p \in \Delta$. (37) is valid also in case $U_k \cap V_0$ is empty, provided that $\epsilon < 1$, since, in this case, $S_{k|0}(p) = 1$.

It follows from $f^\mu{}_{ik} \underset{\mu}{\equiv} f^\mu{}_{ij}f^\mu{}_{jk}$ that

(38) $$f_{ik|\mu} - f_{jk|0}f_{ij|\mu} - f_{ij|0}f_{jk|\mu} \underset{\mu}{\equiv} f^{\mu-1}{}_{ij}f^{\mu-1}{}_{jk} - f^{\mu-1}{}_{ik}.$$

We have

$$f^{\mu-1}{}_{ik} - f^{\mu-1}{}_{ij}f^{\mu-1}{}_{jk} = -(f^{\mu-1}{}_{ij} - f_{ij|0})(f^{\mu-1}{}_{jk} - f_{jk|0})$$
$$+ f^{\mu-1}{}_{ik} - f_{jk|0}f^{\mu-1}{}_{ij} - f_{ij|0}f^{\mu-1}{}_{jk} + f_{ij|0},$$

while $f^{\mu-1}{}_{ik} - f^{\mu-1}{}_{ij}f^{\mu-1}{}_{jk} \underset{\mu-1}{\equiv} 0$. Hence we infer from (31) that

$$f^{\mu-1}{}_{ik}(t) - f^{\mu-1}{}_{ij}(t)f^{\mu-1}{}_{jk}(t) \ll c_1{}^2A(t)^2 \ll (bc_1{}^2/c)A(t)$$

and therefore, by (38),

$$(39) \qquad f_{ik|\mu}(t) - f_{jk|0}f_{ij|\mu}(t) - f_{ij|0}f_{jk|\mu}(t) \ll (bc_1{}^2/c)A(t).$$

This implies that

$$(40) \qquad f_{ji|0}f_{ij|\mu}(t) + f_{ij|0}f_{ji|\mu}(t) \ll (bc_1{}^2/c)A(t),$$

since $f_{ii|\mu}(t) = 0$. Combining (37) and (40) and using (29), we get

$$(41) \qquad f_{ik|\mu}(p,t) \ll ((c_2{}^2c_3/\epsilon) + (bc_2c_1{}^2/c))A(t), \qquad \text{for } p \in U_i \cap U^*{}_k.$$

Now let p be any point of $U_i \cap U_k$. p is contained in one of the neighborhoods, say $U^*{}_j$. If $j = i$ or $j = k$, the estimate of $f_{ik|\mu}(p,t)$ is given already by (37) or (41). If $j \neq i$, $j \neq k$, the estimates of $f_{ij|\mu}(p,t)$ and $f_{jk|\mu}(p,t)$ are given by (41) and (37), respectively, and therefore, using (39), we obtain

$$f_{ik|\mu}(p,t) \ll ((c_2 + c_2{}^3)(c_3/\epsilon) + (1 + c_2{}^2)(bc_1{}^2/c))A(t).$$

We set

$$c_1 = 2(c_2 + 2)(c_2 + c_2{}^3)/\epsilon$$

Clearly $c_2 + 2 > c_3$ if $c > bc_1$. Consequently, if

$$c > 2bc_1(1 + c_2{}^2),$$

we get

$$(c_2 + c_3{}^3)(c_3/\epsilon) + (1 + c_2{}^2)(bc_1{}^2/c) < c_1$$

and therefore

$$f_{ik|\mu}(t) \ll c_1A(t).$$

Combined with (31), this proves (33).

Next, $\psi_{ik|\mu+1}(t) = \psi_{ik|\mu+1}(p,t)$ is defined by

$$\psi_{ik|\mu+1}(t) \underset{\mu+1}{\equiv} f^{\mu}{}_{ik}(t)S^{\mu}{}_k(t) - S^{\mu}{}_i(t).$$

We have

$$f^{\mu}{}_{ik}S^{\mu}{}_k - S^{\mu}{}_i = (f^{\mu}{}_{ik} - f_{ik|0})(S^{\mu}{}_k - S_{k|0})$$
$$+ f_{ik|0}S^{\mu}{}_k + S_{k|0}f^{\mu}{}_{ik} - S^{\mu}{}_i - S_{i|0},$$

while $f^{\mu}{}_{ik}S^{\mu}{}_k - S^{\mu}{}_i \underset{\mu}{\equiv} 0$. Hence it follows from (32) and (33) that

$$f^{\mu}_{ik}(t)S^{\mu}_{k}(t) - S^{\mu}_{i}(t) \ll c_1 A(t)^2 \ll (bc_1/c)A(t).$$

We obtain therefore

$$\psi_{ik|\mu+1}(t) \ll (bc_1/c)A(t). \tag{42}$$

Lemma 2. *We can choose* $\phi_{i|\mu+1}(t) = \phi_{i|\mu+1}(p, t)$ *satisfying*

$$\phi_{i|\mu+1}(p, t) - f_{ik|0}(p)\phi_{k|\mu+1}(p, t) = \psi_{ik|\mu+1}(p, t)$$

such that

$$\phi_{i|\mu+1}(t) \ll (c_4 bc_1/c)A(t), \tag{43}$$

where the constant c_4 *is independent of* μ.

An elementary proof of this lemma will be given in Section 6 below. Clearly (43) implies that

$$S_{i|\mu+1}(t) \ll (bc_1c_4/c)A(t).$$

Consequently, letting

$$c = 2bc_1(1 + c_2^2) + bc_1c_4,$$

we obtain (34).

Now, since the constants b, c, c_1 are independent of μ, we infer by induction on μ that

$$\begin{cases} f_{ik}(t) - f_{ik|0} \ll c_1 A(t), \\ S_i(t) - S_{i|0} \ll A(t). \end{cases} \tag{44}$$

Let $N = \{t \mid \sum_{r=1}^{m} |t_r|^2 < c^2/m\}$. It follows from (44) that the power series $S_i(p, t)$ and $f_{ik}(p, t)$ converge absolutely and uniformly for $t \in N$. Thus $S_i(p, t)$ and $f_{ik}(p, t)$ are holomorphic functions on $U_i \times N$ and $U_i \cap U_k \times N$, respectively, and satisfy (11), (12), (12a), (13). This completes our proof of Theorem 1.

Theorem 2. *Let* $\mathfrak{V} \xrightarrow{\varpi} M$ *be a complex analytic family of compact submanifolds of* W, *where* M *is a spherical domain on the space of* m *complex variables* $t_1, t_2, \cdots, t_m$ *with the center* 0. *If* $\rho_{d,0}\colon T_0 \to H^0(V_0, \Psi_0)$ *is bijective, then* $\rho_{d,t}\colon T_t \to H^0(V_t, \Psi_t)$ *is bijective for each point* t *in a sufficiently small neighborhood* N *of* 0 *on* M.

Proof. Let $\psi_r(t) = \rho_{d,t}(\partial/\partial t_r)$, $r = 1, 2, \cdots, m$. By hypothesis, $\{\psi_1(0), \psi_2(0), \cdots, \psi_m(0)\}$ forms a base of $H^0(V_0, \Psi_0)$. It follows that $\psi_1(t), \psi_2(t), \cdots, \psi_m(t)$ are linearly independent for $t \in N$, while we have

$$\dim H^0(V_t, \Psi_t) \leqq \dim H^0(V_0, \Psi_0), \qquad \text{for } t \in N. \tag{45}$$

Consequently, $\{\psi_1(t), \psi_2(t), \cdots, \psi_m(t)\}$ forms a base of $H^0(V_t, \Psi_t)$ for $t \in N$. This proves our Theorem 2. We remark that the above inequality (45), a special case of the theorem of upper semi-continuity, can easily be derived from the classical theorem to the effect that any family of uniformly bounded holomorphic functions is a normal family.

5. Maximal families; proof of main theorem. First we prove the following

THEOREM 3. *Let* $\mathcal{V} \xrightarrow{\varpi} M$ *be a complex analytic family of compact submanifolds of* W *(of codimension* 1*). If* $\rho_{d,t}: T_t \to H^0(V_t, \Psi_t)$ *is bijective for a point* t *of* M*, then the family* $\mathcal{V} \xrightarrow{\varpi} M$ *is maximal at the point* t.

Proof. Suppose that $M = \{t \mid \sum_{r=1}^{m} | t_r |^2 < 1\}$ and that $\rho_{d,0}: T \to H^0(V_0, \Psi_0)$ is bijective. Moreover, let $\mathcal{V}' \xrightarrow{\varpi'} M'$ be an arbitrary complex analytic family of compact submanifolds of W such that $\varpi'^{-1}(0) = V_0$, where

$$M' = \{s \mid \sum_{r=1}^{l} | s_r |^2 < 1\},$$

and let $N' = \{s \mid \sum_{r=1}^{l} | s_r |^2 < \delta\}$ be a sufficiently small spherical subdomain of M'. Our purpose is to construct a holomorphic map $h: s \to t = h(s)$ of N' into M with $h(0) = 0$ such that $\varpi'^{-1}(s) = \varpi^{-1}(h(s))$.

Let $U_i, w_i(p), \cdots$ have the same meaning as in Section 3 and let $\{S_i(p,t)\}$, $\{f_{ik}(p,t)\}$ be the systems of holomorphic functions $S_i(p,t)$, $f_{ik}(p,t)$, defined respectively on $U_i \times M$, $U_i \cap U_k \times M$, which determine the complex analytic family $\mathcal{V} \subset W \times M$ in the manner described in Section 2. Moreover, let $\{R_i(p,s)\}$, $\{e_{ik}(p,s)\}$ be the corresponding systems which determine $\mathcal{V}' \subset W \times M'$. We have therefore

$$\left\{\begin{aligned} S_i(p,t) &= f_{ik}(p,t) S_k(p,t), \\ R_i(p,s) &= e_{ik}(p,s) R_k(p,s). \end{aligned}\right. \tag{46}$$

In this Section we are exclusively concerned with neighborhoods U_i which meet V_0. Obviously, we may assume that

$$S_i(p,0) = R_i(p,0) = w_i(p),$$

$$f_{ik}(p,0) = e_{ik}(p,0) = f_{ik|0}(p).$$

By means of $S_i(p,t)$ and $R_i(p,s)$, the condition $\varpi'^{-1}(s) = \varpi^{-1}(h(s))$ can be formulated as follows: There exist non-vanishing holomorphic functions $f_i(p,s)$ defined respectively on $U_i \times N'$ such that

$$(47) \qquad f_i(p,s)R_i(p,s) = S_i(p,h(s)), \qquad \text{for } p \in U_i, s \in N'.$$

We write $h(s)$ in the form

$$h(s) = (h_1(s), \cdots, h_r(s), \cdots, h_m(s))$$

and expand each component $h_r(s)$ into power series

$$h_r(s) = \sum_{\mu=1}^{\infty} h_{r|\mu}(s),$$

where $h_{r|\mu}(s)$ is a homogeneous polynomial of degree μ in s. Moreover, let

$$h^\mu{}_r(s) = h_{r|1}(s) + h_{r|2}(s) + \cdots + h_{r|\mu}(s),$$
$$h^\mu(s) = (h^\mu{}_1(s), \cdots, h^\mu{}_r(s), \cdots, h^\mu{}_m(s)).$$

Similarly, let

$$f_i(p,s) = 1 + \sum_{\mu=1}^{\infty} f_{i|\mu}(p,s),$$

where $f_{i|\mu}(p,s)$ is a homogeneous polynomial of degree μ in s, and let

$$f^\mu{}_i(p,s) = 1 + f_{i|1}(p,s) + \cdots + f_{i|\mu}(p,s).$$

For any holomorphic functions $P(s)$, $Q(s)$ in $s = (s_1, s_2, \cdots, s_l)$, we indicate by $P(s) \underset{\mu}{\equiv} Q(s)$ that the power series expansion of $P(s) - Q(s)$ in $s_1, s_2, \cdots, s_l$ contains no terms of degree $\leqq \mu$. Clearly, (47) is equivalent to the system of congruences

$$(48)_\mu \qquad f^\mu{}_i(p,s)R_i(p,s) \underset{\mu}{\equiv} S_i(p,h^\mu(s)), \qquad (\mu = 0, 1, 2, \cdots).$$

In what follows we denote by $\bar{f}^{\mu-1}{}_i(p,s), \bar{R}_i(p,s), \bar{S}_i(p,t), \cdots$ the restrictions of the functions $f^{\mu-1}{}_i(p,s), R_i(p,s), S_i(p,s), \cdots$ to V_0. We expand $S_i(p,t)$ into the power series

$$S_i(p,t) = w_i(p) + S_{i|1}(p,t) + S_{i|2}(p,t) + \cdots$$

and let

$$(49) \qquad S_{i|1}(p,t) = \sum_{r=1}^{m} B_{ir}(p) t_r.$$

The restrictions $\beta_{ir}(p) = \bar{B}_{ir}(p)$ satisfy

$$\beta_{ir}(p) = f_{ik|0}(p)\beta_{kr}(p), \qquad \text{for } p \in V_0 \cap U_i \cap U_k;$$

thus $\{\beta_{ir}(p)\}$ represents an element β_r of $H^0(V_0, \Psi_0)$. In fact, β_r is the infinitesimal displacement $-\rho_{d,0}(\partial/\partial t_r)$. Since, by hypothesis, $\rho_{d,0}$: $T_0 \to H^0(V_0, \Psi_0)$ is bijective, $\{\beta_1, \cdots, \beta_r, \cdots, \beta_m\}$ forms a base of $H^0(V_0, \Psi_0)$.

Now we construct $h^\mu(s)$ and $f^\mu{}_i(p,s)$ satisfying $(48)_\mu$ by induction on μ.

For $\mu=0$ the congruence $(48)_0$ is obvious. Assume therefore that $h^{\mu-1}(s)$ and $f^{\mu-1}{}_i(p,s)$ satisfying $(48)_{\mu-1}$ are already determined. Then we can determine homogeneous polynomials $\Gamma_{i|\mu}(p,s)$ of degree μ in s by

$$(50) \qquad \Gamma_{i|\mu}(p,s) \underset{\mu}{\equiv} f^{\mu-1}{}_i(p,s)R_i(p,s) - S_i(p,h^{\mu-1}(s)).$$

The coefficients of $\Gamma_{i|\mu}(p,s)$ are holomorphic functions in p defined on U_i. The restrictions $\bar{\Gamma}_{i|\mu}(p,s)$ of $\Gamma_{i|\mu}(p,s)$ satisfy

$$(51) \qquad \bar{\Gamma}_{i|\mu}(p,s) = f_{ik|0}(p)\bar{\Gamma}_{k|\mu}(p,s), \qquad \text{for } p \in V_0 \cap U_i \cap U_k.$$

To prove this we remark that

$$(52) \qquad f^{\mu-1}{}_i(s)e_{ik}(s) \underset{\mu-1}{\equiv} f_{ik}(h^{\mu-1}(s))f^{\mu-1}{}_k(s),$$

where we omit the variable p for simplicity. In fact, combinging $(48)_{\mu-1}$ with (46), we obtain

$$\begin{aligned} f^{\mu-1}{}_i(s)e_{ik}(s)R_k(s) &= f^{\mu-1}{}_i(s)R_i(s) \underset{\mu-1}{\equiv} S_i(h^{\mu-1}(s)) \\ &= f_{ik}(h^{\mu-1}(s))S_k(h^{\mu-1}(s)) \underset{\mu-1}{\equiv} f_{ik}(h^{\mu-1}(s))f^{\mu-1}{}_k(s)R_k(s). \end{aligned}$$

This proves (52). Since $\bar{R}_k(s) \underset{0}{\equiv} 0$, we get from (52) the congruence

$$\bar{f}^{\mu-1}{}_i(s)\bar{R}_i(s) = \bar{f}^{\mu-1}{}_i(s)\bar{e}_{ik}(s)\bar{R}_k(s) \underset{\mu}{\equiv} \bar{f}_{ik}(h^{\mu-1}(s))\bar{f}^{\mu-1}{}_k(s)\bar{R}_k(s).$$

Hence we have

$$\bar{f}^{\mu-1}{}_i(s)\bar{R}_i(s) - \bar{S}_i(h^{\mu-1}(s)) \underset{\mu}{\equiv} \bar{f}_{ik}(h^{\mu-1}(s))\{\bar{f}^{\mu-1}{}_k(s)\bar{R}_k(s) - \bar{S}_k(h^{\mu-1}(s))\}.$$

Combining this with (50), we obtain

$$\Gamma_{i|\mu}(p,s) \underset{\mu}{\equiv} \bar{f}_{ik}(p,h^{\mu-1}(s))\bar{\Gamma}_{k|\mu}(p,s) \underset{\mu}{\equiv} \bar{f}_{ik|0}(p)\bar{\Gamma}_{k|\mu}(p,s).$$

This proves (51).

Since $h^{\mu}{}_r(s) = h^{\mu-1}{}_r(s) + h_{r|\mu}(s)$, the congruence $(48)_\mu$ is rewritten in in the form

$$w_i f_{i|\mu}(s) + f^{\mu-1}{}_i(s)R_i(s) \underset{\mu}{\equiv} S_i(h^{\mu-1}(s)) + \sum_{r=1}^{m} B_{ir}(p)h_{r|\mu}(s).$$

We infer therefore that $(48)_\mu$ is equivalent to

$$(53) \qquad \sum_{r=1}^{m} B_{ir}(p)h_{r|\mu}(s) = w_i(p)f_{i|\mu}(p,s) + \Gamma_{i|\mu}(p,s).$$

The restriction of (53) to V_0 gives the equality

$$(54) \qquad \sum_{r=1}^{m} \beta_{ir}(p)h_{r|\mu}(s) = \bar{\Gamma}_{i|\mu}(p,s).$$

Now (51) shows that $\{\bar{\Gamma}_{i|\mu}(p,s)\}$ represents a homogeneous polynomial of degree μ in s *with coefficients in* $H^0(V_0,\Psi_0)$. It follows that there exist homogeneous polynomials $h_{r|\mu}(s)$ of degree μ in s with constant coefficients satisfying (54). We define homogeneous polynomials $f_{i|\mu}(p,s)$ by

$$f_{i|\mu}(p,s)=\{\sum_{r=1}^{m}B_{ir}(p)h_{r|\mu}(s)-\Gamma_{i|\mu}(p,s)\}/w_i(p). \tag{55}$$

We infer readily from (54) that the coefficients of $f_{i|\mu}(p,s)$ are holomorphic functions in p defined on U_i, while it is clear that the $f_{i|\mu}(p,s)$ satisfy (53). This completes our inductive construction of $h^\mu(s)$ and $f^\mu{}_i(p,s)$ satisfying $(48)_\mu$. We note that $h^\mu(s)$ and $f^\mu{}_i(p,s)$ are *uniquely* determined. For our purpose it suffices to show that the power series $h_r(s)$, $f_i(p,s)$ thus determined converge absolutely and uniformly for $s\in N'$, provided that the spherical domain N' *is* sufficiently small.

We prove by induction on μ the estimates

$$h^\mu{}_r(s)\ll A(s), \tag{56$_\mu$}$$

$$f^\mu{}_i(p,s)-1\ll A(s), \tag{57$_\mu$}$$

where

$$A(s)=(b/64c)\sum_{\mu=1}^{\infty}c^\mu(s_1+s_2+\cdots+s_l)^\mu/\mu^2$$

provided that the constants b and c are chosen properly. We have

$$A(s)^2\ll(b/c)A(s)$$

(cf. (28)) and therefore

$$A(s)^\nu\ll(b/c)^{\nu-1}A(s),\qquad\text{for }\nu=2,3,4,\cdots. \tag{58}$$

We choose a constant $a>0$ such that

$$S_i(p,t)-w_i(p)\ll\sum_{\nu=1}^{\infty}a^\nu(t_1+t_2+\cdots+t_m)^\nu, \tag{59}$$

$$R_i(p,s)-w_i(p)\ll\sum_{\nu=1}^{\infty}a^\nu(s_1+\cdots+s_l)^\nu/64\nu^2. \tag{60}$$

Now $(56)_1$ and $(57)_1$ are obvious provided that b is sufficiently large. Assume therefore that $(56)_{\mu-1}$ and $(57)_{\mu-1}$ are already proved. We first estimate

$$\Gamma_{i|\mu}(p,s)\underset{\mu}{\equiv}f^{\mu-1}{}_i(p,s)R_i(p,s)-S_i(p,h^{\mu-1}(s)).$$

Since $\Gamma_{i|\mu}(p,s)$ is a homogeneous polynomial of degree μ in s, the terms $f^{\mu-1}{}_i(p,s)w_i(p)$ and $w_i(p)+S_{i|1}(p,h^{\mu-1}(s))$ contribute nothing to $\Gamma_{i|\mu}(p,s)$.

Hence we get

$$\Gamma_{i|\mu}(p,s) \ll (A(s)+1)\sum_{\nu=1}^{\infty} a^\nu s^\nu/64\nu^2 + \sum_{\nu=2}^{\infty} a^\nu (mA(s))^\nu,$$

where $s^\nu = (s_1 + s_2 + \cdots + s_l)^\nu$. Suppose that

$$c \geqq 2mab, \qquad b > 1.$$

Then we have

$$\sum_{\nu=1}^{\infty} a^\nu s^\nu/64\nu^2 \ll (a/64c)\sum_{\nu=1}^{\infty} c^\nu s^\nu/\nu^2 = (a/b)A(s)$$

and therefore, by (58),

$$(A(s)+1)\sum_{\nu=1}^{\infty} a^\nu s^\nu/64\nu^2 \ll ((a/c)+(a/b))A(s).$$

Moreover, using (58), we get

$$\sum_{\nu=2}^{\infty} a^\nu (mA(s))^\nu \ll A(s)\sum_{\nu=2}^{\infty} a^\nu m^\nu b^{\nu-1}/c^{\nu-1} \ll (2m^2a^2b/c)A(s).$$

Thus we obtain

$$\Gamma_{i|\mu}(p,s) \ll c_5 A(s), \tag{61}$$

where

$$c_5 = (a/c) + (a/b) + (2m^2a^2b/c).$$

In view of (54), there exists a constant κ which is independent of μ such that (61) implies

$$h_{r|\mu}(s) \ll \kappa c_5 A(s).$$

Finally, since the coefficients of $f_{i|\mu}(p,s)$ are holomorphic on the polycylinder U_i defined by $|w_i(p)| < 1$, $|z^1_i(p)| < 1, \cdots, |z^n_i(p)| < 1$, we infer from (55) that

$$f_{i|\mu}(p,s) \ll (ma\kappa + 1)c_5 A(s)$$

(note that (59) implies $|B_{ir}(p)| < a$). Consequently, by choosing the constants b and c properly, we obtain

$$h_{r|\mu}(s) \ll A(s),$$
$$f_{i|\mu}(p,s) \ll A(s).$$

This completes our inductive proof of $(56)_\mu$ and $(57)_\mu$, and the convergence of $h_r(s)$, $f_i(p,s)$ for $s \in N'$ follows.

Now it is clear that our main theorem follows from Theorems 1, 2 and 3.

6. Proof of Lemma 2. For simplicity, we write U_i for $V_0 \cap U_i$ and denote by $\mathfrak{U}$ the covering $\{U_i\}$ of V_0. For any 0-cochain $\phi = \{\phi_i(p)\}$, 1-cochain $\psi = \{\psi_{ik}(p)\}, \cdots$ on $\mathfrak{U}$ with coefficients in the sheaf Ψ_0, we define the norms of $\phi, \psi, \cdots$ by

$$\| \phi \| = \max_i \sup_{p \in U_i} | \phi_i(p) |,$$

$$\| \psi \| = \max_{i,k} \sup_{p \in U_i \cap U_k} | \psi_{ik}(p) |,$$

$$\cdots.$$

The coboundary $\delta\phi$ of ϕ is defined by

$$(\delta\phi)_{ik}(p) = f_{ik|0}(p)\phi_k(p) - \phi_i(p), \qquad p \in U_i \cap U_k.$$

Our purpose is to show the existence of a constant c_4 having the following properties: *If ψ is the coboundary of a 0-cochain, then we can find a 0-cochain ϕ with $\delta\phi = \psi$ such that*

$$\| \phi \| \leqq c_4 \| \psi \|. \tag{62}$$

For any ψ which is the coboundary of a 0-cochain, we define

$$\iota(\psi) = \inf_{\delta\phi = \psi} \| \phi \|.$$

It suffices to prove the existence of a constant c such that

$$\iota(\psi) \leqq c \| \psi \|. \tag{63}$$

Assume that such a constant c does not exist. Then we find a sequence $\psi', \psi'', \cdots, \psi^{(\mu)}, \cdots$ such that

$$\iota(\psi^{(\mu)}) = 1, \qquad \| \psi^{(\mu)} \| < 1/\mu.$$

The equality $\iota(\psi^{(\mu)}) = 1$ implies that there exists $\phi^{(\mu)}$ with $\delta\phi^{(\mu)} = \psi^{(\mu)}$ satisfying

$$\| \phi^{(\mu)} \| < 2.$$

We take a covering $\{U^*_i\}$ of V_0 by *compact* subsets $U^*_i \subset U_i$. Moreover, we write each $\phi^{(\mu)}$ explicitly in the form $\{\phi_i^{(\mu)}(p)\}$. Since $| \phi_k^{(\mu)}(p) | < 2$ for $p \in U_k$, there exists a subsequence $\phi^{(\mu_1)}, \phi^{(\mu_2)}, \cdots, \phi^{(\mu_\nu)}$ of $\phi', \phi'', \cdots$ such that $\phi_k^{(\mu_\nu)}(p)$ converges absolutely and uniformly on U^*_k for each k. On the other hand, it follows from $\| \delta\phi^{(\mu)} \| = \| \psi^{(\mu)} \| < 1/\mu$ that

$$| f_{ik|0}(p)\phi_k^{(\mu)}(p) - \phi_i^{(\mu)}(p) | < 1/\mu, \qquad \text{for } p \in U_i \cap U_k. \tag{64}$$

Since any point $p \in U_i$ is covered by at least one U^*_k, we infer from (64)

Hence we get

$$\Gamma_{i|\mu}(p,s) \ll (A(s)+1)\sum_{\nu=1}^{\infty} a^\nu s^\nu/64\nu^2 + \sum_{\nu=2}^{\infty} a^\nu (mA(s))^\nu,$$

where $s^\nu = (s_1 + s_2 + \cdots + s_l)^\nu$. Suppose that

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In view of (54), there exists a constant κ which is independent of μ such that (61) implies

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Finally, since the coefficients of $f_{i|\mu}(p,s)$ are holomorphic on the polycylinder U_i defined by $|w_i(p)| < 1$, $|z^1_i(p)| < 1, \cdots, |z^n_i(p)| < 1$, we infer from (55) that

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$$\| \phi \| = \max_{i} \sup_{p \in U_i} | \phi_i(p) |,$$

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The coboundary $\delta\phi$ of ϕ is defined by

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Our purpose is to show the existence of a constant c_4 having the following properties: *If ψ is the coboundary of a 0-cochain, then we can find a 0-cochain ϕ with $\delta\phi = \psi$ such that*

$$\| \phi \| \leqq c_4 \| \psi \|. \tag{62}$$

For any ψ which is the coboundary of a 0-cochain, we define

$$\iota(\psi) = \inf_{\delta\phi = \psi} \| \phi \|.$$

It suffices to prove the existence of a constant c such that

$$\iota(\psi) \leqq c \| \psi \|. \tag{63}$$

Assume that such a constant c does not exist. Then we find a sequence $\psi', \psi'', \cdots, \psi^{(\mu)}, \cdots$ such that

$$\iota(\psi^{(\mu)}) = 1, \qquad \| \psi^{(\mu)} \| < 1/\mu.$$

The equality $\iota(\psi^{(\mu)}) = 1$ implies that there exists $\phi^{(\mu)}$ with $\delta\phi^{(\mu)} = \psi^{(\mu)}$ satisfying

$$\| \phi^{(\mu)} \| < 2.$$

We take a covering $\{U^*_i\}$ of V_0 by *compact* subsets $U^*_i \subset U_i$. Moreover, we write each $\phi^{(\mu)}$ explicitly in the form $\{\phi_i^{(\mu)}(p)\}$. Since $| \phi_k^{(\mu)}(p) | < 2$ for $p \in U_k$, there exists a subsequence $\phi^{(\mu_1)}, \phi^{(\mu_2)}, \cdots, \phi^{(\mu_\nu)}$ of $\phi', \phi'', \cdots$ such that $\phi_k^{(\mu_\nu)}(p)$ converges absolutely and uniformly on U^*_k for each k. On the other hand, it follows from $\| \delta\phi^{(\mu)} \| = \| \psi^{(\mu)} \| < 1/\mu$ that

$$| f_{ik|0}(p)\phi_k^{(\mu)}(p) - \phi_i^{(\mu)}(p) | < 1/\mu, \qquad \text{for } p \in U_i \cap U_k. \tag{64}$$

Since any point $p \in U_i$ is covered by at least one U^*_k, we infer from (64)

that $\phi_i^{(\mu_\nu)}(p)$ converges absolutely and uniformly on the whole neighborhood U_i. Let $\phi_i(p) = \lim_\nu \phi_i^{(\mu_\nu)}(p)$ and let $\phi = \{\phi_i(p)\}$. Then we have

$$\| \phi^{(\mu_\nu)} - \phi \| \to 0 \qquad (\nu \to \infty),$$

while it follows from (64) that $\delta\phi = 0$ and therefore $\delta(\phi^{(\mu_\nu)} - \phi) = \psi^{(\mu_\nu)}$. This contradicts with $\iota(\psi^{(\mu_\nu)}) = 1$.

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AN EXAMPLE TO A PROBLEM OF ABHYANKAR.*

By Masayoshi Nagata.

In the paper of Abhyankar, "On the field of definition of a non-singular birational transform of an algebraic surface," *Annals of Mathematics*, vol. 65, No. 2 (1957), the following question was asked:

Let K be a function field of dimension 2 over an imperfect ground field k of characteristic p $(\neq 0)$ and let k' be a purely inseparable extension of k of degree p. Let K' be the field generated by K over k'. Let R be a normal spot of K over k and let R' be the derived normal ring of R in K'. Assume that R' is a regular local ring. Is then R regular?

In the present note, we shall show an example where R is not regular. In fact, we shall give an example of such pair (R, R') with the following additional conditions:

$\mathfrak{m}$ and $\mathfrak{m}'$ being the maximal ideals of R and R' respectively, (i) $\mathfrak{m}R' = \mathfrak{m}'$ and (ii) $[R'/\mathfrak{m}' : R/\mathfrak{m}] = p^2$ $(> [K' : K])$.

Let k be a field of characteristic $p \neq 0$ which has elements u and v such that $[k(u^{1/p}, v^{1/p}) : k] = p^2$. Let x and y be variables over k and let z be a root of the polynomial $Z^p + yZ + u + v^{1/p}x$. Set $w = v^{1/p}x + zy$. We shall show that the pair of $R = k[x, y, w]_{(x,y,w)}$ and $R' = k(v^{1/p})[x, y, z]_{(x,y)}$ (observe that $Z^p + u$ is irreducible over $k(v^{1/p})$ and therefore x and y generate a maximal ideal of $k(v^{1/p})[x, y, z]$) is the required example.

Let K and K' be the field of quotients of R and R' respectively. Then we see that $K' = K(v^{1/p})$. R' is obviously a regular local ring. Since z modulo $(x, y) = u^{1/p}$, the residue class field of R' is $k(u^{1/p}, v^{1/p})$. Therefore we see that the condition (ii) above is satisfied by R and R'; (i) is obviously satisfied. Therefore we have only to prove the normality of R.

Since $w = v^{1/p}x + zy$, we have $z = (w - v^{1/p})/y$. Since $z^p + yz + u + v^{1/p}x = 0$, i.e., $z^p + u + w = 0$, we have $w^p + y^p w + (uy^p - vx^p) = 0$, which is

* Received May 12, 1958.

the irreducible monic equation for w over $k[x,y]$. Therefore, by the mixed Jacobian criterion for simplicity due to Zariski, we see that R is the only one singular spot of the affine model defined by $k[x,y,w]$, which shows the normality of R. Thus the proof is completed.

Remark. If we want to construct similar examples without requiring that K is a regular extension of k, then the following construction gives a much simpler example than above:

With the same k, u, v, x and y as above, set $z=u^{1/p}x+v^{1/p}y$. Then $R=k[x,y,z]_{(x,y,z)}$ and $R'=k(u^{1/p},v^{1/p})[x,y]_{(x,y)}$ give a required pair.

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ON THE UNIQUENESS OF THE CAUCHY PROBLEM FOR PARABOLIC EQUATIONS.* [1]

By Avner Friedman.

Introduction. Consider the second order parabolic equation

$$\partial u/\partial t = Lu \tag{0.1}$$

in the strip

$$0 < t < d, \qquad x = (x_1, \cdots, x_n), \qquad -\infty < x_i < \infty. \tag{0.2}$$

Tychonoff [13] constructed a solution of (0.1) which vanishes on $t = 0$ but is not identically zero; thus there is no uniqueness of the Cauchy problem. However, if a solution $u(x,t)$ of (0.1) satisfies the growth condition

$$u(x,t) = O(\exp\{K\,|\,x\,|^2\}) \qquad (K \geqq 0) \tag{0.3}$$

and if it vanishes on $t = 0$, then it vanishes identically in the strip (0.2). The proof of this theorem for the heat equation was given by Tychonoff [13]. It was extended to general second order parabolic equations by Krzyzański [5] (see also [6]). It was then extended to parabolic systems of order $2m$ provided (0.3) is replaced by

$$u(x,t) = O(\exp\{K\,|\,x\,|^{2m/(2m-1)}\}) \qquad (K \geqq 0). \tag{0.4}$$

In the case where the coefficients depend only on t it was proved by Ladyzhenskaya [7] (for $K = 0$ it was proved by Petrowski [9]). In the general case it was proved by Eidelman [4; p. 73] (see also [3]). Slobodetski [12] announced that if the coefficients of the parabolic system depend only on t, then uniqueness holds under the assumption

$$\int\int \exp\{-K\,|\,x\,|^{2m/(2m-1)}\}\,|\,u(x,t)\,|\,dxdt < \infty \qquad (K > 0) \tag{0.5}$$

which is weaker than the assumption (0.4). (The integration in (0.5) is taken over the strip (0.2).)

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Another type of assumption sufficient to ensure uniqueness was obtained by Widder [14]. He proved in case of the heat equation that a solution satisfying

$$u(x,t) \geqq 0 \tag{0.6}$$

and vanishing for $t=0$ must vanish identically. Assumption (0.6) seems to be more natural than assumption (0.3) since temperatures are always nonnegative. Serrin [11] announced an extension of Widder's result to solutions of the equation

$$u_t = a(x)u_{xx} + b(x)u_x + c(x)u \tag{0.7}$$

with Hölder continuous and uniformly bounded coefficients and with $a(x) \geqq \text{const.} > 0$.

In this paper we extend all the above-mentioned results. We first consider general second order parabolic equations and prove uniqueness (i) under the assumption (0.5) with $m=1$, and (ii) under the assumption (0.6). Then we consider general parabolic systems and furnish the tools for the proof of the uniqueness of the Cauchy problem under the assumption (0.5).

1. Statement of uniqueness theorems for second order equations. Consider the equation

$$\partial u/\partial t = Lu \equiv \sum_{i,j=1}^{n} a_{ij}(x,t)\partial^2 u/\partial x_i\partial x_j + \sum_{i=1}^{n} b_i(x,t)\partial u/\partial x_i + c(x,t)u, \tag{1.1}$$

where $x=(x_1,\cdots,x_n)$ varies in the whole n-dimensional Euclidean space E_n and $0<t<d$. Denote by D the topological product of E_n with the interval $0<t<d$. We shall make on L the following assumptions:

(A) L is uniformly elliptic in $\bar{D}$ (the closure of D), that is, there exists a positive constant K such that for all $(x,t)\in\bar{D}$ and for every real vector $\xi=(\xi_1,\cdots,\xi_n)$

$$\sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j \geqq K\sum_{i=1}^{n}\xi_i^2. \tag{1.2}$$

(B) The functions

$$a_{ij},\ (\partial/\partial x_\lambda)a_{ij},\ (\partial^2/\partial x_\lambda\partial x_\mu)a_{ij},\ (\partial/\partial t)a_{ij},\ b_i,\ (\partial/\partial x_\lambda)b_i,\ c \tag{1.3}$$

are locally Hölder continuous and uniformly bounded in $\bar{D}$; denote by K' a bound on these functions.

By a solution of (1.1) in D is meant a function $u(x,t)$ which is con-

tinuous in $\bar{D}$ and which has continuous partial derivatives $\partial u/\partial x_i$, $\partial^2/\partial x_i\partial x_j$, $\partial u/\partial t$ in D satisfying (1.1).

THEOREM 1. *Let L satisfy the assumptions* (A), (B). *If $u(x,t)$ is a solution of* (1.1) *in the strip D which satisfies the growth condition*

$$\int_0^d \int_{E_n} \exp\{-H_0 |x|^2\} |u(x,t)| \, dxdt < \infty \qquad (H_0 > 0) \tag{1.4}$$

and if $u(x,0) \equiv 0$ for $x \in E_n$, then $u(x,t) \equiv 0$ in D.

THEOREM 2. *Let L satisfy the assumptions* (A), (B). *If $u(x,t)$ is a nonnegative solution of* (1.1) *in the strip D and if $u(x,0) \equiv 0$ for $x \in E_n$, then $u(x,t) \equiv 0$ in D.*

2. Proof of Theorem 1. We shall make use of the fundamental solution $\Gamma(x,t;\xi,\tau)$ $(t > \tau)$ of (1.1) defined in the whole strip $\bar{D}$, which was constructed by Dressel [2] under the assumptions (A), (B). As a function of (x,t) it satisfies the equation $\partial\Gamma/\partial t = L\Gamma$ and as a function of (ξ,τ) it satisfies the adjoint equation $\partial\Gamma/\partial\tau = -L^*\Gamma$, where

$$L^*v = \sum_{i,j=1}^{n} (\partial^2/\partial x_i\partial x_j)(a_{ij}v) - \sum_{i=1}^{n} (\partial/\partial x_i)(b_i v) + cv.$$

Γ can be written in the form

$$\Gamma(x,t;\xi,\tau) = Z(x,t;\xi,\tau)[1 + O((t-\tau)^{\frac{1}{2}})] \tag{2.1}$$

(making use of the explicit form of Γ and of [2; Lemma 2]), where

$$Z(x,t;\xi,\tau) = \begin{cases} [F(\xi,\tau)(t-\tau)^{n/2}]^{-1}\exp\{-\sigma(x,t;x-\xi)/4(t-\tau)\} & \text{if } t > \tau \\ 0 \text{ if } t < \tau. \end{cases} \tag{2.2}$$

Here, $\sigma(x,t;x-\xi) = \sum A_{ij}(x,t)(x_i-\xi_i)(x_j-\xi_j)$, (A_{ij}) is the matrix inverse to (a_{ij}), and $F(\xi,\tau)$ is a certain positive function depending on σ (see [1; p. 191]).

Later on we shall need the fact that $\Gamma(x,t;\xi,\tau) > 0$ if $t > \tau$. For $t-\tau$ sufficiently small this follows from (2.1). Noting now that $\Gamma(x,t;\xi,\tau) \to 0$ as $|x| \to \infty$, uniformly with respect to t, and recalling that $\Gamma(x,\tau+\epsilon,\xi,\tau) > 0$ if ϵ is positive and sufficiently small, we apply the maximum principle [8] and thus conclude that $\Gamma(x,t;\xi,\tau)$ is a positive function for $t > \tau$. We finally mention that for any $i = 1,\cdots,n$,

$$|\partial\Gamma(x,t;\xi,\tau)/\partial x_i| = |\partial Z(x,t;\xi,\tau)/\partial x_i| [1 + O((t-\tau)^{\frac{1}{2}})]; \tag{2.3}$$

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the constants in $O((t-\tau)^{\frac{1}{2}})$ both in (2.1) and in (2.3) depend only on K, K'.

It is enough (in both Theorems 1 and 2) to prove that $u(x,t)\equiv 0$ for $x\in E_n$, $0\leqq t<\eta$ for some positive η, since then we can carry out the same argument step by step. Later on whe shall make use of the fact that η may be taken to be sufficiently small.

Let $(\bar{x},\bar{t})$ be an arbitrary fixed point in the strip $0<t<\eta$. If we prove that $u(\bar{x},\bar{t})=0$ then, by the previous remark, the proof of Theorem 1 is completed. Denote by B_R the sphere in E_n with center $\bar{x}$ and radius R and denote by B'_R the domain $B_{R+1}-B_R$. Let $h(\xi)$ be a twice continuously differentiable function in E_n satisfying the following properties:

$$(2.4)\qquad h(\xi)=\begin{cases}1 & \text{if } 0\leqq|\xi|\leqq R\\ 0 & \text{if } R+1\leqq|\xi|<\infty\end{cases}$$

$$(2.5)\qquad 0\leqq h(\xi)\leqq 1 \text{ and } \sum_i|\partial h(\xi)/\partial\xi_i|+\sum_{i,j}|\partial^2h(\xi)/\partial\xi_i\partial\xi_j|\leqq A \quad \text{for } R<|\xi|<R+1,$$

where A is an appropriate universal constant.

Integrating Green's identity

$$(2.6)\qquad \begin{aligned} &v(Lw-\partial w/\partial\tau)-w(L^*v+\partial v/\partial\tau)\\ &=\sum_{i=1}^n(\partial/\partial\xi_i)[\sum_{k=1}^n(va_{ik}\partial w/\partial\xi_k-wa_{ik}\partial v/\partial\xi_k-vw\partial a_{ik}/\partial\xi_k)+b_iwv]\\ &-(\partial/\partial\tau(wv)\end{aligned}$$

with $w=u(\xi,\tau)$, $v=h(\xi)\Gamma(\bar{x},\bar{t};\xi,\tau)$ over the whole strip $0<\tau<\bar{t}-\epsilon(\epsilon>0)$ and taking $\epsilon\to 0$ we get, on using the properties of h, Γ and the assumption $u(\xi,0)\equiv 0$,

$$(2.7)\qquad u(\bar{x},\bar{t})=\int_0^{\bar{t}}\int_{B'_R}u[L^*(h\Gamma)+\partial(h\Gamma)/\partial\tau]d\xi d\tau.$$

Since $L^*(h\Gamma)+\partial(h\Gamma)/\partial\tau$ involves only linear combinations of Γ and $\partial\Gamma/\partial\xi_i$ with coefficients which (by (2.5) and assumption (B)) are bounded by a constant independent of R, we can easily estimate the right side of (2.7), making use of (2.1), (2.2), (2.3). We get

$$(2.8)\qquad |u(\bar{x},\bar{t})|\leqq A\exp\{-HR^2/\eta\}\int_0^{\bar{t}}\int_{B'_R}|u(\xi,\tau)|\,d\xi d\tau,$$

where A, H are positive constants depending only on K, K'.

We now note that (1.4) implies

$$(2.9)\qquad \int_0^{\bar{t}}\int_{E_n}\exp\{-2H_0|\xi-\bar{x}|^2\}\,|u(\xi,\tau)|\,d\xi d\tau<\infty.$$

Hence,

$$\int_0^{\bar t}\int_{B'_R}\exp\{-2H_0\,|\,\xi-\bar x\,|^2\}\,|\,u(\xi,\tau)\,|\,d\xi d\tau\to 0 \text{ as } R\to\infty. \tag{2.10}$$

Taking η to satisfy $H/\eta > 2H_0$, then letting $R\to\infty$ in (2.8) and using (2.10), we conclude that $u(\bar x,\bar t)=0$ and the proof is thereby completed.

3. Proof of Theorem 2. Without loss of generality we may assume that $c(x,t)\leqq 0$.

Consider the function

$$U_R(x,t)=\int_{|\xi|<R}\Gamma(x,t;\xi,\tau)u(\xi,\tau)\,d\xi, \tag{3.1}$$

where Γ is the fundamental solution of (1.1) in D, introduced in § 2. By [2] we conclude that

$$\lim_{t\to\tau+0}\ \sup_{x\to x^0} U_R(x,t)\leqq\begin{cases}u(x^0,\tau) & \text{if } |\,x^0\,|\leqq R\\ 0 & \text{if } |\,x^0\,|>R.\end{cases}$$

Hence, the function

$$v(x,t)=u(x,t)-U_R(x,t) \tag{3.2}$$

satisfies

$$\liminf v(x,t)\geqq 0 \tag{3.3}$$

as $t\to\tau+0$, $x\to x^0$. Furthermore, from the form of Γ we conclude that $U_R(x,t)\to 0$ as $|\,x\,|\to\infty$, uniformly with respect to t, $0\leqq t\leqq d$. Hence, (3.3) holds also when $|\,x\,|\to\infty$, uniformly with respect to t, $0\leqq t\leqq d$. Using the maximum principle [8], we easily conclude that $v(x,t)\geqq 0$ in D; more explicitly,

$$\int_{|\xi|<R}\Gamma(x,t;\xi,\tau)u(\xi,\tau)\,d\xi\leqq u(x,t) \text{ for } x\in E_n,\ t>\tau. \tag{3.4}$$

Noting that the left side of (3.4) is monotone increasing in R and taking $R\to\infty$, we get

$$\int_{E_n}\Gamma(x,t;\xi,\tau)u(\xi,\tau)\,d\xi\leqq u(x,t) \text{ for } x\in E_n, t>\tau \tag{3.5}$$

(the existence of the integral is implied).

Integrating both sides of (3.5) with respect to τ, $0<\tau<\eta$ and taking $x=0$, we obtain

$$\int_0^\eta\int_{E_n}\Gamma(0,t;\xi,\tau)u(\xi,\tau)\,d\xi d\tau\leqq\eta u(0,t). \tag{3.6}$$

Let η, t $(0<\eta<t<d)$ be sufficiently small numbers for which

$$(3.7)\qquad \Gamma(0,t;\xi,\tau)\geqq A\exp\{-H\,|\,\xi\,|^2\} \text{ for all } 0\leqq\tau<\eta \quad (A>0, H>0).$$

By (2.1), (2.2), such numbers exist, and A, H depend only on K, K', t and η. Substituting (3.7) into (3.6), we get

$$(3.8)\qquad A\int_0^\eta\int_{E_n}\exp\{-H\,|\,\xi\,|^2\}u(\xi,\tau)\,d\xi d\tau\leqq\eta u(0,t)<\infty.$$

Thus, u satisfies the growth assumption of Theorem 1 in the strip $0<\tau<\eta$. Applying Theorem 1, we get $u(\xi,\tau)\equiv 0$ for $\xi\in E_n$, $0\leqq\tau<\eta$. We can now proceed step by step to prove that $u\equiv 0$ in D.

Remark 1. Using the estimates

$$(3.9)\qquad \partial\Gamma(x,t;\xi,\tau)/\partial x^i,\ \partial^2\Gamma(x,t;\xi,\tau)/\partial x^i\partial x^j,\ \partial\Gamma(x,t;\xi,\tau)/\partial t$$
$$=O(\Gamma(x,2t;\xi,2\tau)) \qquad (i=1,2)$$

(as $|\,x-\xi\,|\to\infty$), it follows that one may differentiate the left side of (3.5) twice with respect to x and once with respect to t and that the operations of integration and differentiation are permutable. Hence, the left side of (3.5) is a nonnegative solution of (1.1). Since it takes on $t=\tau$ the same values as $u(x,t)$, we can apply Theorem 2 to the nonnegative solution

$$(3.10)\qquad w(x,t)=u(x,t)-\int_{E_n}\Gamma(x,t;\xi,\tau)u(\xi,\tau)\,d\xi.$$

We conclude that $w\equiv 0$ in D, that is,

$$(3.11)\qquad u(x,t)=\int_{E_n}\Gamma(x,t;\xi,\tau)u(\xi,\tau)\,d\xi.$$

The representation (3.11) *holds for all nonnegative solutions of* (1.1) *in th strip* $0<t<d$. Using this representation, one can easily derive a Harnack type inequality for solutions of (1.1) which are nonnegative in the whole strip D. We shall consider here one example.

Assume that $a_{ij}(x,t)\equiv\delta_{ij}$ and let $u(x,t)$ be a positive solution in the strip $0\leqq t\leqq\eta$. If η is sufficiently small, then, by (2.1), (2.2),

$$A_1\leqq\Gamma(x,t;\xi,\tau)/(t-\tau)^{-n/2}\exp\{-\,|\,x-\xi\,|^2/4(t-\tau)\}\leqq A_2,$$

where A_1, A_2 are positive constants. Hence, by (3.11),

$$A_1v(x,t)\equiv A_1\int_{E_n}(t-\tau)^{-n/2}\exp\{-\,|\,x-\xi\,|^2/4(t-\tau)\}u(\xi,\tau)\,d\xi\leqq u(x,t)$$
$$\leqq A_2\int_{E_n}(t-\tau)^{-n/2}\exp\{-\,|\,x-\xi\,|^2/4(t-\tau)\}u(\xi,\tau)\,d\xi\equiv A_2v(x,t),$$

where $v(x,t)$ is a solution of the heat equation. We fix $\tau=0$. We can now apply the Harnack inequality derived by Resch [10] for positive solutions of the heat equation (with $n=1$, but which is easily extended to general n), and thus concludes:

If $0<t'<t<\eta$, $x\in E_n$, $x'\in E_n$, then

$$u(x,t)\geqq (A_1/A_2)H(x,t)u(x',t'), \tag{3.12}$$

where

$$H(x,t)=(t'/t)^{n/2}\exp\{-|x-x'|^2/4(t-t')\}. \tag{3.13}$$

In particular, if $|x'-x|^2<A_3(t-t')$ (A_3 positive constant), then

$$u(x,t)\geqq A_4u(x',t'), \qquad A_4 \text{ positive constant.} \tag{3.14}$$

Remark 2. From the proof of Theorem 2 one can easily see that if $u(x,t)$ satisfies (1.4) and if $u(x,0)\geqq 0$ for all $x\in E_n$, then for all $(\bar{x},\bar{t})$ in the strip $0<\bar{t}<\eta$ (η sufficiently small, depending on K, K' and H_0),

$$u(\bar{x},\bar{t})\geqq \int_{E_n}\Gamma(\bar{x},\bar{t};\xi,0)u(\xi,0)d\xi.$$

Thus, $u(x,t)$ is nonnegative in the strip $0\leqq t\leqq\eta$. Continuing step by step, we find that $u(x,t)$ is nonnegative in the whole strip D. Thus, *if $u(x,t)$ satisfies* (1.4) *and $u(x,0)\geqq 0$ for $x\in E_n$, then $u(x,t)\geqq 0$ in D and the representation* (3.11) *holds in D.*

4. Uniqueness theorems for general systems. In this section we consider the uniqueness of the Cauchy problem for general parabolic systems of the form

$$\partial u_i/\partial t=\sum_{j=1}^{N}\sum_{\Sigma k_s\leqq 2m} A^{ij}{}_{k_1\cdots k_n}(x,t)\,\partial^{k_1+\cdots+k_n}u_j/\partial x_1{}^{k_1}\cdots\partial x_n{}^{k_n} \qquad (i=1,\cdots,N) \tag{4.1}$$

defined in the domain D of § 1. The system (4.1) can be written in the matrix form

$$\begin{aligned}\partial U/\partial t &= P(x,t,\tfrac{1}{2\pi i}\partial/\partial x)U\\ &= P_0(x,t,\tfrac{1}{2\pi i}\partial/\partial x)U+P_1(x,t,\tfrac{1}{2\pi i}\partial/\partial x)U,\end{aligned} \tag{4.2}$$

where P_0 is the principal part and P_1 involves only derivatives of orders smaller than $2m$. We shall need the following assumptions:

(A′) The system (4.1) is parabolic in D (in the sense of Petrowski), that is, there exists a positive number δ such that for all (x,t) in D and for

any real vector $\sigma = (\sigma_1, \cdots, \sigma_n)$, the characteristic roots $\lambda = \lambda(\sigma)$ of the matrix $P_0(x, t, \sigma)$ satisfy the inequality $\operatorname{Re}\{\lambda\} < -\delta$.

(B′) The coefficients of the k-th derivatives which appear on the right side of (4.1) have derivatives of the first $k+1$ orders with respect to x, all being uniformly continuous with respect to (x, t) in D.

The assumptions (A′), (B′) are sufficient to ensure the existence of a fundamental solution to the system adjoint to (4.1) (see Eidelman [4; p. 71]). Denote it by $Z(x, t; \xi, \tau)$ $(t > \tau)$. As a function of (ξ, τ) it satisfies the system adjoint to (4.2). It also satisfies the inequalities

$$(4.3)\qquad |(\partial/\partial\xi^i) Z(x, t; \xi, \tau)| \leqq \text{const.}/(t-\tau)^{(n+i)/2m}$$
$$\exp\{-\text{const.}\, |x-\xi|^{2m/(2m-1)}/(t-\tau)^{1/(2m-1)}\} \quad (i = 0, 1, \cdots, 2m),$$

where the constants are positive. Finally, for any continuous function $\phi(x, t)$ defined in $\bar{D}$ and for any bounded domain G in E_n,

$$(4.4)\qquad \lim_{\tau \to t-0} \int_G Z(x, t; \xi, \tau) \phi(\xi, \tau)\, d\xi = \phi(x, t).$$

Using these results, one can easily modify the proof of Theorem 1 and derive the following uniqueness theorem.

THEOREM 3. *Let the system* (4.1) *satisfy the assumptions* (A′), (B′). *If* $(u_1(x, t), \cdots, u_N(x, t))$ *is a solution of the system* (4.1) *in the strip* D *which satisfies the assumption*

$$(4.5)\qquad \int_0^d \int_{E_n} \exp\{-H_0 |x|^{2m/(2m-1)}\} |u_i(x, t)|\, dx dt < \infty$$
$$(i = 1, \cdots, N; H_0 > 0)$$

and if $u_i(x, 0) \equiv 0$ $(i = 1, \cdots, N)$ *for* $x \in E_n$, *then* $u_i(x, t) \equiv 0$ *for* $x \in E_n$, $0 \leqq t \leqq d$.

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A RELATION BETWEEN CW-COMPLEXES AND FREE c. s. s. GROUPS.*

By Daniel M. Kan.

1. Introduction. The homotopy theory of c. s. s. complexes which satisfy the extension condition, the homotopy theory of CW-complexes, and the loop homotopy theory of free c. s. s. groups are equivalent ([9], [5]). While c. s. s. complexes are completely combinatorial, a considerable advantage of CW- complexes is the fact that a given homotopy type can often be represented by a very small model. For instance an n-sphere may be represented by a CW-complex with two cells (one in dimension 0 and one in dimension n), while any c. s. s. complex satisfying the extension condition of the same homotopy type contains infinitely many non-degenerate simplices in all dimensions $\geqq n$. Of course the n-sphere may be represented by a c. s. s. complex which does not satisfy the extension condition containing only two non-degenerate simplices (in the dimension 0 and n), but it can readily be seen that for a two cell CW-n-sphere with an $(n+1)$-cell attached by a map of degree q this (i. e. representing it by a c. s. s. complex with three non-degenerate simplices in dimensions 0, n and $n+1$) cannot be done for q sufficiently large.

Free c. s. s. groups are a fortiori c. s. s. complexes (which even satisfy the extension condition). However, as was remarked in [8], Remark 5. 6, they also very much behave like CW-complexes. Like a CW-complex which is determined by its cells and their attaching maps, a free c. s. s. group is determined (see § 2) by a suitable subset (the elements of which are called generators) together with an attaching element for every generator.

It is the purpose of this note to show that this similarity is not accidental, but that for every CW-complex K (with only one 0-cell) one may construct a free c. s. s. group B which has the loop homotopy type of the loop space on K and which has as many generators in dimension $n-1$ as K has cells in dimension n, and conversely. This implies that for any homotopy type equally small models exists among free c. s. s. groups as among CW-complexes. For instance if K is the above two cell CW-n-sphere with an $(n+1)$-cell attached

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by a map of degree q, then the free c. s. s. group which has a generator α in dimension $n-1$ and a generator β in dimension n with attaching element α^q has the loop homotopy of the loop space on K.

There are two chapters and an appendix. Chapter I contains several propositions which illustrate the similarity of CW-complexes and free c. s. s. groups, while Chapter II deals with the exact relationship between them. In the Appendix, which is more or less independent of the rest of the paper, we consider the operations union, cone and suspension and their analogues for c. s. s. groups.

Free use will be made of the notation and the results of [4] and [5].

Chapter I. Similarity of CW-complexes and free c. s. s. groups.

2. CW-bases. The CW-behaviour of free c. s. s. groups can easiest be described using a special kind of bases, therefore called CW-bases. We recall the definition and the main properties.

Definition 2.1. Let F be a free c. s. s. group ([5], Definition 5.1). A subset $\mathfrak{F} \subset F$ will be called a *CW-basis* of F if

(a) $\mathfrak{F}_n = \mathfrak{F} \cap F_n$ freely generates F_n for all $n \geqq 0$,

(b) $\mathfrak{F}$ is closed under degeneracies, i. e. $\sigma \in \mathfrak{F}_n$ implies $\sigma\eta^i \in \mathfrak{F}_{n+1}$ for all $0 \leqq i \leqq n$,

(c) if $\sigma \in \mathfrak{F}_n$ *is* non-degenerate, then $\sigma\epsilon^i = e_{n-1}$ (e_{n-1} = unit of F_{n-1}) for all $0 \leqq i < n$.

The non-degenerate elements of $\mathfrak{F}$ are called *generators*; for a generator $\sigma \in \mathfrak{F}_n$ the elements $\sigma\epsilon^n \in F_{n-1}$ will be called the *attaching element* of σ.

The following proposition was proved in [8], § 5.

PROPOSITION 2.2. *Every free c. s. s. group has a CW-basis.*

It is easily seen that a free c. s. s. group has more than one CW-basis. In going from one CW-basis to another, Propositions 2.5 and 2.6 below will be useful. In order to formulate them we need:

Notational convention 2.3. Let F be a c. s. s. group and let $\mathfrak{F} \subset F$ be a set which is closed under degeneracies (Definition 2.1(b)). For every non-degenerate element $\sigma \in \mathfrak{F} \cap F_n$ we will then denote by $\{\mathfrak{F} - \sigma\}$ the set obtained from $\mathfrak{F}$ by omitting σ and all its degeneracies; and if $\tau \in F_n, \not\subset \mathfrak{F}$

is non-degenerate, then $\{\mathfrak{F}+\tau\}$ will denote the set obtained from $\mathfrak{F}$ by adding τ and all its degeneracies.

Definition 2.4. For every integer $n \geqq 0$, the *n-skeleton* F^n of a c. s. s. group F is the c. s. s. subgroup $F^n \subset F$ generated by F_n, i. e. the smallest c. s. s. subgroup containing F_n. By the (-1)*-skeleton* F^{-1} we mean the c. s. s. subgroup generated by the element e_0.

PROPOSITION 2.5. *Let $\mathfrak{F}$ be a CW-basis of the free c. s. s. group F. If $\sigma \in \mathfrak{F}_n$ is a generator and $\tau \in F_n$ is in subgroup generated by $\mathfrak{F}_n - \sigma$ and is such that $\tau\epsilon^i = e_{n-1}$ for $0 \leqq i < n$, then $\{\mathfrak{F} - \sigma + \sigma\tau\}$ is a CW-basis of F.*

The proof of this proposition is straightforward.

An immediate consequence is:

PROPOSITION 2.6. *If $\sigma \in \mathfrak{F}_n$ is a generator and $\tau \in (F^{n-1})_n$ is such that $\tau\epsilon^i = e_{n-1}$ for $0 \leqq i < n$, then $\{\mathfrak{F} - \sigma + \sigma\tau\}$ is a CW-basis of F.*

3. Some similarities. It is clear from the definition of a *CW*-basis that a free c. s. s. group is completely determined by the generators of a *CW*-basis together with their attaching elements, just as *CW*-complexes ([13]) are determined by their cells and attaching maps. This analogy may be carried further. If to a *CW*-complex a cell is attached by two different attaching maps, which are homotopic, then the resulting complexes have the same homotopy type. A slightly stronger result holds for free c. s. s. groups: attaching a generator by two different attaching elements which are homotopic in the sense of [4], § 2, yields free c. s. s. groups which are not only of the same homotopy type, but are even isomorphic. For the exact formulation we need:

Definition 3.1. Let K be a *CW*-complex, $L \subset K$ a subcomplex, c a cell of $K - L$ and λ its attaching map. Then K is said to be *obtained from L by attaching c by the attaching map λ*, if $K = L \cup c$ (Notation $K = L \cup_\lambda c$ or $K = L \cup c$).

Definition 3.2. Let F be a free c. s. s. group, $A \subset F$ a c. s. s. subgroup and $\sigma \in F_n - A_n$ a non-degenerate n-simplex. Then F is said to be *obtained from A by attaching σ by attaching element $\alpha = \sigma\epsilon^n$* if there exists a *CW*-basis $\mathfrak{A}$ of A such that $\{\mathfrak{A} + \sigma\}$ is a *CW*-basis of F (Notation $F = A *_\alpha \sigma$ or $F = A * \sigma$). It is an immediate consequence of Proposition 2.5 that if one *CW*-basis $\mathfrak{A}$ of A is such that $\{\mathfrak{A} + \sigma\}$ is a *CW*-basis of F, then every *CW*-basis of A has this property.

PROPOSITION 3.3. *Let $K = L \cup_\lambda c$ and $M = N \cup_\nu d$, where* $\dim c = \dim d$, *and let $f: L \to N$ be a homeomorphism such that $f \circ \lambda \sim \nu$. Then f may be extended to a homotopy equivalence $f': K \to M$.*

This well known proposition follows immediately from the definition of a CW-complex ([13]. We now state its analogue.

PROPOSITION 3.4. *Let $F = A *_\alpha \sigma$ and $G = B *_\beta \tau$, where* $\dim \sigma = \dim \tau = n$, *let $h: A \to B$ be an isomorphism and let $\rho \in B_n$ be such that $\rho: h\alpha \sim \beta$ ([4], Definition 2.2). Then h may be extended to an isomorphism $h': F \to G$ such that $h'\sigma = \tau \cdot \rho^{-1} \cdot \rho\epsilon^{n-1}\eta^{n-1}$.*

Proof. The definition of the relation $\sim$ ([4], § 2) implies that

$$(\rho^{-1} \cdot \rho\epsilon^{n-1}\eta^{n-1})\epsilon^i = e_{n-1}, \qquad 0 \leqq i < n,$$

$$(\rho^{-1} \cdot \rho\epsilon^{n-1}\eta^{n-1})\epsilon^n = \beta^{-1} \cdot h\alpha$$

and hence in view of Definition 3.2 and the freeness of F there exists exactly one c. s. s. homomorphism $h': F \to G$ such that $h' \mid A = h$ and

$$h'\sigma = \tau \cdot \rho^{-1} \cdot \rho\epsilon^{n-1}\eta^{n-1}.$$

Let $\mathfrak{A}$ be a CW-basis of A, then $\mathfrak{F} = \{\mathfrak{A} + \sigma\}$ is a CW-basis of F. Furthermore $\{h\mathfrak{A} + \tau\}$ is a CW-basis of G and in view of Proposition 2.5 so is $\mathfrak{G} = \{h\mathfrak{A} + \tau \cdot \rho^{-1} \cdot \rho\epsilon^{n-1}\eta^{n-1}\}$. That h' is an isomorphism now follows from the fact that h' induces a one to one correspondence between the elements of $\mathfrak{F}$ and those of $\mathfrak{G}$.

Another "well known" proposition on CW-complexes and its analogue for free c. s. s. groups are:

PROPOSITION 3.5. *Let $K = L \cup_\lambda c$ and $M = N \cup_\nu d$, where* $\dim c = \dim d$, *and let $f: L \to N$ be a homotopy equivalence such that $f \circ \lambda = \nu$. Then the map $f': K \to M$ given by $f' \mid L = f$ and $f' \circ \sigma_c = \sigma_d$ (where σ_c and σ_d denote the characteristic maps of c and d) is a homotopy equivalence.*

PROPOSITION 3.6. *Let $F = A *_\alpha \sigma$ and $G = B *_\beta \tau$, where* $\dim \sigma = \dim \tau$, *and let $h: A \to B$ be a loop homotopy equivalence ([5], §§ 3, 4) such that $h\alpha = \beta$. Then the c. s. s. homomorphism $h': F \to G$ given by $h' \mid A = h$ and $h'\sigma = \tau$ is a loop homotopy equivalence.*

Proof of Proposition 3.5. Let C denote the mapping cylinder of f ([3], p. 108), let $l: L \to C$ and $n: N \to C$ be the injections and $p: C \to N$ the projection and denote by

$$l': K \to C \cup_{l \circ \lambda} c', \qquad p': C \cup_{l \circ \lambda} c' \to M$$

$$n': M \to C \cup_{n \circ \nu} d', \qquad p'': C \cup_{n \circ \nu} d' \to M$$

the continuous maps such that

$$\begin{array}{ll} l' \mid L = l & l' \circ \sigma_c = \sigma_{c'} \\ n' \mid N = n & n' \circ \sigma_d = \sigma_{d'} \\ p' \mid C = p & p' \circ \sigma_{c'} = \sigma_d \\ p'' \mid C = p & p'' \circ \sigma_{d'} = \sigma_d \end{array}$$

where σ_c denotes the characteristic map of c, etc. Because $l(L)$ and $n(N)$ are strong deformation retracts of C ([3]), it follows readily that l' and n' are homotopy equivalences. Clearly p'' is a homotopy inverse of n' and hence p'' is also a homotopy equivalence. By Proposition 3.3 the identity map $i: C \to C$ may be extended to a homotopy equivalence $i': C \cup c' \to C \cup d'$ and it is readily seen that i' may be chosen in such a manner that $p' \sim p'' \circ i'$. Hence p' is a homotopy equivalence and the proposition now follows from the fact that $f' = p' \circ l'$.

Proof of Proposition 3.6. The proof of Proposition 3.6 is completely analogous to that of Proposition 3.5, using loop homotopy ([5], §§3, 4) instead of homotopy and Proposition 3.4 instead of Proposition 3.3; the analogue of the mapping cylinder is the free c. s. s. group obtained from $(I \otimes A) * B$ by identifying for every simplex $\sigma \in A$

$$(\epsilon_1{}^0\eta^0 \cdot \cdot \cdot \eta^{(\dim \sigma)-1} \otimes \sigma) \text{ with } h\sigma.$$

The details are left to the reader.

Chapter II. Relation between *CW*-complexes and free c. s. s. groups.

4. Twisting functions. In relating a *CW*-complex and a free c. s. s. group we need the following two intermediate notions.

(a) The first Eilenberg subcomplex of the total singular complex of a topological space with base point ([1]).

(b) J. C. Moore's notion of a twisting function, relating a c. s. s. complex and a c. s. s. group ([11]).

We briefly recall the definition of the latter and give some of its properties.

Definition 4.1. Let S be a *reduced* c.s.s. complex (i.e. S has only one 0-simplex) and let B be a c.s.s group. A *twisting function* $t\colon S \to B$ then is a function which lowers dimension by one and is such that for every integer $n > 0$ and every simplex $\sigma \in S_n$

$$(t\sigma)\epsilon^i = t(\sigma\epsilon^i) \qquad 0 \leqq i < n-1,$$

$$(t\sigma)\epsilon^{n-1} = t(\sigma\epsilon^{n-1}) \cdot (t(\sigma\epsilon^n)^{-1})$$

$$(t\sigma)\eta^i = t(\sigma\eta^i) \qquad 0 \leqq i \leqq n-1,$$

$$e_n = t(\sigma\eta^n).$$

The notion of a twisting function is closely related with the construction G of [4], which assigns to a reduced c.s.s. complex S a free c.s.s. group GS which has the homotopy type of the loops on S. We recall its definition: For every integer $n \geqq 0$, G_nS is a group which has

(i) one generator $\bar{\sigma}$ for every $(n+1)$-simplex $\sigma \in S_{n+1}$,

(ii) one relation $\overline{\tau\eta^n} = e_n$ for every n-simplex $\tau \in S_n$.

The face and degeneracy homomorphism are given by

$$\bar{\sigma}\epsilon^i = \overline{\sigma\epsilon^i} \qquad 0 \leqq i < n$$

$$\bar{\sigma}\epsilon^n = \overline{\sigma\epsilon^n} \cdot (\overline{\sigma\epsilon^{n+1}})^{-1}$$

$$\bar{\sigma}\eta^i = \overline{\sigma\eta^i} \qquad 0 \leqq i \leqq n.$$

The following propositions express the close relationship between the construction G and twisting functions. Their proofs are straightforward.

PROPOSITION 4.2. *Let S be a reduced c.s.s. complex, let B be a c.s.s. group and let $t\colon S \to B$ be a twisting function. Then there exists one c.s.s. homomorphism $gt\colon GS \to B$ such that for every simplex $\sigma \in S$*

$$(gt)\bar{\sigma} = t\sigma$$

PROPOSITION 4.3. *The function g of Proposition 4.2 sets up a one to one correspondence between*

(a) *the twisting functions $S \to B$,*

(b) *the c.s.s. homomorphisms $GS \to B$.*

With a twisting function $t\colon S \to B$ one may associate ([11], [4]) a principal fibre bundle with S as base and B as fibre. If the total complex of

this bundle is contractible then following J. W. Milnor ([10], [4]) we call B a *loop complex* of S (under t). This notion of being a loop complex may also be defined using the construction G instead of twisting functions. For the case that B is free this is done in the following proposition.

PROPOSITION 4.4. *Let S be a reduced c.s.s. complex, let B be a free c.s.s. group and let t: t: $S \to B$ be a twisting function. Then B is a loop complex complex of S under t* ([4], *Definition* 6.3) *if and only if the map gt: $GS \to B$ is a loop homotopy equivalence.*

The proof of this proposition is similar to that of [5], Theorem 11.2.

5. The main relation. In this section we shall define a relation between CW-complexes with one 0-cell and free c.s.s. groups and state its main properties. In order to simplify the argument only CW-complexes of which the characteristic maps of a special kind (called reduced CW-complexes) will be considered.

Let Δ_n denote an Euclidean n-simplex with vertices $A_0, \cdots, A_n$ and let $\Lambda_{n-1} \subset \Delta_n$ be the union of all its faces except the one opposite A_{n-1}. Then we define

Definition 5.1. A CW-complex K will be called *reduced* if

(a) K contains only one 0-cell p,

(b) for every integer $n > 0$ and every n-cell $c \in K$, the characteristic map is a map σ_c: $\Delta_n \to K$ such that $\sigma_c(\Lambda_{n-1}) = p$.

Definition 5.2. Let K be a reduced CW-complex, p its only 0-cell, and denote by $S(K)$ the first Eilenberg subcomplex of its total singular complex ([1]), i.e. an n-simplex of $S(K)$ is any continuous map σ: $\Delta_n \to K$ such that $\sigma(A_i) = p$ for all i. Let B be a free c.s.s. group and let t: $S(K) \to B$ be a twisting function (Definition 4.1). Then we will say that t is *regular* if

(a) the elements $t\sigma_c$ (where c runs through the cells of $K - p$) are distinct and form the generators of a CW-basis of B,

(b) for every subcomplex $L \subset K$,

$$t(S(L)) \subset B(L),$$

where $B(L) \subset B$ denotes the c.s.s. subgroup generated ([4], § 5) by the elements $t\sigma_c$ for which $c \in L$.

An immediate consequence of this definition is:

PROPOSITION 5.3. *If $t\colon S(K) \to B$ is regular, then so is $t \mid S(L)\colon S(L) \to B(L)$ for every subcomplex $L \subset K$ and in particular $t \mid S(K^n)\colon S(K^n) \to B^{n-1}$ for all $n \geqq 0$.*

The following example is a rather trivial one; in several proofs it will, however, be used as a starting point for induction.

Example 5.4. Let K consist of one point, let B have only one element in every dimension and let $t\colon S(K) \to B$ be the only such twisting function. Then clearly t is regular. Moreover the unique map $gt\colon GS(K) \to B$ (§ 4) is an isomorphism.

The main properties of regular twisting functions may be summed up in the following theorems.

THEOREM 5.5. *If $t\colon S(K) \to B$ is regular, then B is a loop complex of $S(K)$ under t (§ 4 and [4], § 6). I.e. there exists a principal bundle with basis $S(K)$, fibre B, twisting function t and a contractible total complex.*

COROLLARY 5.6. *If $t\colon S(K) \to B$ is regular, then for every subcomplex $L \subset K$, $B(L)$ is a loop complex of $S(L)$ under $t \mid S(L)$; in particular B^{n-1} is a loop complex of $S(K^n)$ under $t \mid S(K^n)$ for all $n \geqq 0$.*

COROLLARY 5.7. *Let ϕ be the only 0-simplex of $S(K)$. Then the boundary homomorphisms*

$$\partial^n\colon \pi_i(S(K^n); \phi) \to \pi_{i-1}(B^{n-1}; e_0)$$

of the fibre sequence associated with the principal bundle

$$((S(K^n); \phi), B^{n-1}, t \mid S(K^n))$$

([4], §§ 3, 6), *are isomorphisms for all i and n.*

Proof of Theorem 5.5. It suffices (Proposition 4.4) to show that the map $gt\colon GS(K) \to B$ is a loop homotopy equivalence. This is done by induction on the cells of K.

Order the cells of K in such a manner that $\dim c < \dim d$ implies $c < d$ and denote by $L_c \subset K$ the subcomplex consisting of all cells d with $d < c$. Let $K_c = L_c \cup c$ and suppose it has already been shown that $g(t \mid S(K_d))$ is a loop homotopy equivalence for $d < c$; then clearly $g(t \mid S(L_c))$ is so. Let $Q \subset S(K_c)$ be the subcomplex consisting of $S(L_c)$ and the simplex σ_c and its degeneracies and let $j\colon Q \to S(K_c)$ be the inclusion map. Then (Proposition 3.6) the composition $g(t \mid S(K_c)) \circ Gj\colon GQ \to B(K_c)$ is a loop homotopy equivalence. The natural map $p(L_c)\colon |S(L_c)| \to L_c$ of the geo-

metric realization of $S(L_c)$ onto L_c is a homotopy equivalence ([9]) and so is (Proposition 3.5) the composition $p(K_c)\circ|j|\colon|Q|\to K_c$. As $p(K_c)$ is also a homotopy equivalence, it follows that j is a weak homotopy equivalence. [5], § 11 now yields that Gj is a loop homotopy equivalence and hence so is $g(t\mid S(K_c))$. This proves the induction step.

In order to be able to start the induction we must show that $g(t\mid S(K^0))\colon GS(K^0)\to B^{-1}$ is a loop homotopy equivalence. This is example 5.4.

That gt is a loop homotopy equivalence now follows by induction.

THEOREM 5.8. *Let $t\colon S(K)\to B$ be regular and let $\mathfrak{B}$ denote the CW-basis of B consisting of the elements $t\sigma_c$ and their degeneracies. Then*

(a) *t induces a one to one correspondence between the cells of K and the geneartors of $\mathfrak{B}$ one dimension lower,*

(b) *for every integer $n>0$ and every n-cell $c\in K$ we have*

$$\partial^{n-1}\alpha_c=\beta_c,$$

where $\alpha_c\in\pi_{n-1}(S(K^{n-1});\phi)$ denotes the element containing the attaching map of the cell c and $\beta_c\in\pi_{n-2}(B^{n-2};e_0)$ is the element containing the attaching element $(t\sigma_c)\epsilon^{n-1}$ of the corresponding generator $t\sigma_c$ of $\mathfrak{B}$.

Proof. Part (a) is a restatement of definition 5.2(a), while part (b) follows immediately from the facts that the attaching map of c is homotopic with $\sigma_c\epsilon^{n-1}$ and that $t(\sigma_c\epsilon^{n-1})=(t\sigma_c)\epsilon^{n-1}$ and from the definition of the boundary homomorphism ∂^{n-1} ([4], § 3).

Remark 5.9. The definition of a regular twisting function could be slightly weakened by replacing condition 5.2(b) by

(b′) $$t(S(K^n))\subset B^{n-1}\text{ for all }n\geqq 0.$$

In this case the second half of Proposition 5.3, Theorem 5.5, the second half of Corollary 5.7 and Theorem 5.8 remain true.

6. Existence theorems. It will be shown that every reduced CW-complex is related to a free c.s.s. group by a regular twisting function and conversely. In order to prove this we need the following theorem which shows how, starting from a CW-complex and free c.s.s. group which are related, one may obtain another such pair.

THEOREM 6.1. *Let $K=L\cup_\lambda c$ be a reduced CW-complex and let $B=C*_\alpha\sigma$ be a free c.s.s. group, where* $\dim c=1+\dim\sigma=n$. *Let*

$s\colon S(L) \to C$ be a regular twisting function and suppose that $\partial^{n-1}\{\lambda\} = \{\alpha\}$, i. e. there exists an element $\rho \in C_{n-1}$ such that $\rho\colon s(\sigma_c\epsilon^{n-1}) \sim \alpha$ ([4], Definition 2.2). Then there exists a twisting function $t\colon S(K) \to B$ such that

(a) *t is regular,*

(b) *$t \mid S(L) = s$,*

(c) *$t\sigma_c = \sigma \cdot \rho^{-1} \cdot \rho\epsilon^{n-2}\eta^{n-2}$.*

Proof. Let $Q \subset S(K)$ be the subcomplex consisting of $S(L)$ and σ_c and its degeneracies. Then (see the proof of Theorem 5.5) the inclusion map $j\colon Q \to S(K)$ induces a loop homotopy equivalence $Gj\colon GQ \to GS(K)$. Similarly for every subcomplex $M \subset K$ the intersection $Gj \cap GS(M)\colon GQ \cap GS(M) \to GS(M)$ is a loop homotopy equivalence and iterated application of the loop homotopy extension theorem ([7], § 6) yields the existence of a loop homotopy inverse $h\colon GS(K) \to GQ$ of Gj such that

(i) $h \mid GQ$ is the identity,

(ii) $h(GS(M)) \subset GS(M)$ for every subcomplex $M \subset K$.

As $\rho\colon s(\sigma_c\epsilon^{n-1}) \sim \alpha$ it follows by application of Propositions 3.4 and 3.6 that the map $gs\colon GS(L) \to C$ may be extended to a loop homotopy equivalence $k\colon GQ \to B$ such that $k\bar{\sigma}_c = \sigma \cdot \rho^{-1} \cdot \rho\epsilon^{n-2}\eta^{n-2}$. Let $t\colon GS(K) \to B$ be the unique twisting function (Proposition 4.3) such that $gt = k \circ h$. A straightforward computation then yields that t has all the desired properties.

THEOREM 6.2. *For every reduced CW-complex K there exist a free c.s.s. group B and a regular twisting function $t\colon S(K) \to B$.*

The proof of Theorem 6.2 is similar to that of Theorem 6.3; induction is used on the cells of K as in the proof of Theorem 5.5. The details are left to the reader.

THEOREM 6.3. *For every free c.s.s. group B there exist a reduced CW-complex K and a regular twisting function $t\colon S(K) \to B$.*

Proof. Let $\mathfrak{B}$ be a CW-basis of B. The proof then goes by induction on the generators of $\mathfrak{B}$. Order the generators of $\mathfrak{B}$ in such a manner that $\dim \sigma < \dim \tau$ implies $\sigma < \tau$, let $C_\sigma \subset B$ be the c.s.s. subgroup generated by the generators τ for which $\tau < \sigma$ and let $B_\sigma = C_\sigma * \sigma$. Suppose for all $\tau < \sigma$ a reduced CW-complex K_τ and a regular twisting function $t_\tau\colon S(K_\tau) \to B_\tau$ have already been defined such that for $\tau < \rho < \sigma$, K_τ is a subcomplex of K_ρ and $t_\tau = t_\rho \mid S(K_\tau)$. Let $L_\sigma = \bigcup_{\tau<\sigma} K_\tau$ and $s_\sigma = \bigcup_{\tau<\sigma} t_\sigma$; then clearly s_σ:

$S(L_\sigma) \to C_\sigma$ is regular. If $B_\sigma = C_\sigma *_\alpha \sigma$, $\dim \sigma = n-1$, define K_σ by $K_\sigma = L_\sigma \cup_\lambda . c$, i.e. by attaching a cell c to L_σ, where $\dim c = n$ and its attaching map λ is such that $\partial^{n-1}\{\lambda\} = \{\alpha\}$. This is possible in view of Corollary 5.7. Theorem 6.1 now yields an extension $t_\sigma: S(K_\sigma) \to B_\sigma$ of s_σ which is regular. This proves the induction step. The theorem now follows by induction starting from Example 5.4.

7. Homotopy uniqueness theorems. The question in how far a related CW-complex and free c.s.s. group determine each other will be answered in Theorems 7.1 and 7.2 below. Two CW-complexes related to the same free c.s.s. group are clearly of the same homotopy type, but the other way around a slightly stronger statement may be made: two free c.s.s. groups related to the same CW-complex are not only of the same loop homotopy type, but are even isomorphic.

We now give the exact formulation.

THEOREM 7.1. *Let $t: S(K) \to B$ and $s: S(L) \to C$ be regular twisting functions. Then for every continuous map $f: K \to L$ there exist c.s.s. homomorphisms $a: B \to C$ such that diagram 7.1a is commutative up to a loop homotopy; any two such maps are loop homotopic. And for every c.s.s. map $a: B \to C$ there exist continuous maps $f: K \to L$ such that diagram 7.1a is commutative up to a loop homotopy; any two such maps are homotopic.*

7.1a
$$\begin{array}{ccc} GS(K) & \xrightarrow{GS(f)} & GS(L) \\ \downarrow{\scriptstyle gt} & & \downarrow{\scriptstyle gs} \\ B & \xrightarrow{a} & C \end{array}$$

THEOREM 7.2. *Let $t: S(K) \to B$ and $s: S(K) \to C$ be regular twisting functions. Then there exist isomorphisms $f: B \to C$ such that diagram 7.2a is commutative up to a loop homotopy. Any two such isomorphisms are loop homotopic.*

(7.2a)
$$\begin{array}{ccc} & GS(K) & \\ {\scriptstyle gt}\swarrow & & \searrow{\scriptstyle gs} \\ B & \xrightarrow{f} & C \end{array}$$

Remark 7.3. If the definition of a regular twisting function is weakened as in Remark 5.9, then Theorems 6.1, 6.2, 6.3, 7.1 and 7.2 remain true.

Proof of Theorem 7.1. The second half of the theorem follows immediately from the fact that gs and gt are loop homotopy equivalences (Proposition 4.4 and Theorem 5.5), while the first half is a consequence of [5], §§ 9 and 11.

For the proof of Theorem 7.2 we need:

PROPOSITION 7.4. *Let $t\colon S(K)\to B$ and $s\colon S(K)\to C$ be regular, let $K=L\cup c$ and let $f(L)\colon B(L)\to C(L)$ be an isomorphism such that the following diagram is commutative up to a loop homotopy*

(7.4a)
$$\begin{tikzcd} & GS(L) \arrow[dl, "g(t \mid S(L))"'] \arrow[dr, "g(s \mid S(L))"] & \\ B(L) \arrow[rr, "f(L)"] & & C(L) \end{tikzcd}$$

Then $f(L)$ may be extended to an isomorphism $f\colon B\to C$ such that diagram 7.2a *is commutative up to a loop homotopy.*

Proof of Theorem 7.2. The second half of the theorem follows immediately from the fact that gs and gt are loop homotopy equivalences (Proposition 4.4 and Theorem 5.5). The proof of the first half goes by induction on the cells of K as in the proof of Theorem 5.5, starting from Example 5.4.

For every cell $c\in K$ let K_c, $L_c\subset K$ be as in the proof of Theorem 5.5 and suppose that for every cell $e<c$ an isomorphism $f(K_e)\colon B(K_e)\to C(K_e)$ has already been defined such that diagram 7.4a, with K_e instead of L, is commutative up to a loop homotopy and such that for $e<d<c$, $f(K_e)=f(K_d)\mid B(K_e)$. Let $f(L_c)=\bigcup_{e<c} f(K_e)$ then clearly $f(L_c)\colon B(L_c)\to C(L_c)$ is an isomorphism such that diagram 7.4a, with L_c instead of L, is commutative up to a loop homotopy. Proposition 7.4 yields an extension $f(K_c)$ of $f(L_c)$ which is an isomorphism and is such that diagram 7.4a, with K_c instead of L, is commutative up to a loop homology, which proves the induction step.

Proof of Proposition 7.4. Let $Q\subset S(K)$ be as in the proof of Theorem 6.1. Then the injection $Gj\colon GQ\to GS(K)$ is a loop homotopy equivalence and hence it suffices to show that $f(L)$ may be extended to an isomorphism $f\colon B\to C$ such that the following diagram is commutative up to a loop homotopy.

$$\begin{tikzcd} & GQ \arrow[dl, "g(t \mid Q)"'] \arrow[dr, "g(s \mid Q)"] & \\ B \arrow[rr, "f"] & & C \end{tikzcd}$$

Let $h(L): I \otimes GS(L) \to C(L)$ be a loop homotopy

$$h(L): g(s \mid S(L)) \sim (f(L) \circ g(t \mid S(L))),$$

then by the loop homotopy extension theorem ([7], § 6) $h(L)$ may be extended to a loop homotopy $h: g(s \mid Q) \sim k$. A straightforward computation, using Proposition 2.5, shows that h may be chosen such that $C = C(L) * k\bar{\sigma}_c$. As $B = B(L) * t\sigma_c$ and

$$f(L)((t\sigma_c)\epsilon^{n-1}) = f(L)(t(\sigma_c\epsilon^{n-1})) = (f(L) \circ gt)\bar{\sigma}_c\epsilon^{n-1} = k(\bar{\sigma}_c\epsilon^{n-1})$$

it follows (Proposition 3.4) that $f(L)$ admits an extension to an isomorphism $f: B \to C$ such that $f(t\sigma_c) = k\bar{\sigma}_c$. The proposition now follows from the fact that $k = f \circ g(t \mid Q)$ and $k \sim g(s \mid Q)$.

Appendix.

8. Union and free product. It will be shown (Theorems 8.1 and 8.3) that the free product of two c. s. s. groups is the analogue of the notion of union, with the base points identified, for CW-complxes or c. s. s. complexes.

For two reduced CW-complexes K and K' let $K \vee K'$ denote their union with identification of the 0-cels and let $B * B'$ denote the free product of the c. s. s. groups B and B'. Then we have

THEOREM 8.1. *For any two regular twisting functions $t: S(K) \to B$ and $t': S(K') \to B'$, there exists a regular twisting function*

$$s: S(K \vee K') \to B * B'.$$

Proof. This follows immediately from Theorem 6.1 using induction on the cells of K or on those of K'.

For two reduced c. s. s. complexes S and S' let $S \vee S'$ denote their union with identification of the 0-simplices and their degeneracies. Then ([4], Corollary 20.2)

THEOREM 8.2. *If S and S' are reduced c. s. s. complexes, then*

$$G(S \vee S') = GS * GS'$$

The analogue of Theorem 8.1 for c. s. s. complexes then is:

THEOREM 8.3. *Let the free c.s.s. groups B and B' be loop complexes ([4], § 6) of the reduced c.s.s. complexes S and S' under the twisting functions t and t' respectively. Then $B * B'$ is a loop complex of $S \vee S'$ under the twisting function t'': $S \vee S' \rightarrow B * B'$ given by*

$$t''\sigma = t\sigma, \qquad \sigma \in S, \qquad\qquad t''\sigma = t'\sigma, \qquad \sigma \in S'.$$

Proof. By Proposition 4.4 gt and gt' are loop homotopy equivalences, and so is ([6], Theorem 5.3) the map $gt * gt'$: $GS * GE' \rightarrow B * B'$. The theorem now follows from Theorem 8.2, Proposition 4.4 and the fact that $gt'' = gt * gt'$.

9. The cone. A construction C will be described which assigns to a free c. s. s. group B a free c. s. s. group CB and it will be shown that this construction is the analogue of the reduced cone constructions for CW-complexes and c. s. s. complexes. This construction resembles the construction W of Eilenberg-MacLane ([12]); it uses free products instead of dircet products.

Definition 9.1. The *cone* CB of a c. s. s. group B is the c. s. s. group defined as follows. For every integer $n \geqq 0$

$$C_nB = B_n * B_{n-1} * \cdot \cdot \cdot * B_0.$$

For every integer $k \geqq 0$ and every element $\sigma \in B_n$ let $(k,\sigma) \in C_{n+k}B$ denote the image of σ under the injection $B_n \rightarrow C_{n+k}B$. The face and degeneracy homomorphisms ϵ^i: $C_nB \rightarrow C_{n-1}B$ and η^i: $C_n \rightarrow C_{n+1}B$ then are given by the formulas

$$\text{(9.1a)} \quad \begin{array}{lll} (k,\sigma)\epsilon^i = (k,\sigma\epsilon^{i-k}), & (k,\sigma)\eta^i = (k,\sigma\eta^{i-k}), & i \geqq k \\ (k,\sigma)\epsilon^i = (k-1,\sigma), & (k,\sigma)\eta^i = (k+1,\sigma), & i < k. \end{array}$$

Similarly for a c. s. s. homomorphism f: $B \rightarrow B'$ we define a c. s. s. homomorphism Cf: $CB \rightarrow CB'$ by

$$Cf(k,\sigma) = (k, f\sigma)$$

PROPOSITION 9.2. *If B is a free c.s.s. group, then so is CB.*

Proof. Let $\mathfrak{B}$ be a basis of B ([8], Definition 2.1) and let $\mathfrak{A} \subset CB$ consist of the elements $(0,\sigma)$ and $(1,\sigma)$ and their degeneracies, where σ runs through the generators of $\mathfrak{B}$. It then follows easily from Definition 9.1 that $\mathfrak{A}$ is a basis of CB.

For a reduced CW-complex K denote by CK the *reduced cone* of K, i.e. the complex obtained from $I \times K$ by shrinking to a point of the subcomplex $(1 \times K) \cup (I \times K^0)$. CK may of course be supposed to be reduced. Then we have

THEOREM 9.3. *For every regular twisting function $t\colon S(K) \to B$ there exists a regular twisting function $t'\colon S(CK) \to CB$.*

This is proved by induction on the cells of CK, using Theorem 6.1. The details are left to the reader.

We recall a definition of reduced cone for c.s.s. complexes

Definition 9.4. Let S be a reduced c.s.s. complex and ϕ its only 0-simplex. The *reduced cone* of S is the c.s.s. complex CS defined as follows. For every integer $n \geqq 0$ the n-simplices of CS are the pairs (k, σ), where $k \geqq 0$ is an integer and $\sigma \in S_{n-k}$ is a simplex, identifying $(k, \phi\eta^0 \cdots \eta^{n-k-1})$ with (n, ϕ). The face and degeneracy operators are determined by the formulas 9.1a.

A simple computation yields

THEOREM 9.5. *Let S be a reduced c.s.s. complex. Then the c.s.s. homomorphism $h\colon GCS \to CGS$ given by*

$$h(\overline{k, \sigma}) = (k, \bar{\sigma}) \qquad \sigma \in S,$$

is an isomorphism.

And of course we have

THEOREM 9.6. *For every c.s.s. group B, CB is contractible.*

Proof. The formulas 9.1a imply that the function $d\colon CB \to CB$ given by $d(k, \sigma) = (k+1, \sigma)$ is a contracting homotopy, i.e. $(d\tau)\epsilon^i = d(\tau\epsilon^{i-1})$ for $i > 0$ and $(d\tau)\epsilon^0 = \tau$. From this it readily follows that CB is contractible.

10. The suspension. By "collapsing" the cone construction C of 9.1 new construction $\bar{C}$ may be obtained which is the analogue of the reduced suspension. This construction resembles the $\bar{W}$-construction of Eilenberg-MacLane ([12]).

Definition 10.1. The *suspension* $\bar{C}B$ of a c.s.s. group B is the c.s.s. group obtained from CB by adding for every integer $n \geqq 0$ and every simplex $\sigma \in B_n$ a relation $(0, \sigma) = e_n$. This clearly implies

$$\bar{C}_n B = B_{n-1} * B_{n-2} * \cdots * B_0$$

For a c. s. s. homomorphism $f\colon B \to B'$, $\bar{C}f\colon \bar{C}B \to \bar{C}B'$ will denote the c. s. s. homomorphism induced by $Cf\colon CB \to CB'$.

PROPOSITION 10.2. *If B is a free c. s. s. group, then so is $\bar{C}B$.*

The proof is similar to that of Proposition 9.2.

For a reduced CW-complex K denote by $\bar{C}K$ the *reduced suspension* of K, i. e. the complex obtained from $I \times K$ by shrinking to a point the subcomplex $(0 \times K) \cup (1 \times K) \cup (I \times K^0)$. $\bar{C}K$ may of course be supposed to be reduced. Then

THEOREM 10.3. *For every regular twisting function $t\colon S(K) \to B$ there exists a regular twisting function $t'\colon S(\bar{C}K) \to \bar{C}B$.*

This is proved by induction on the cells of $\bar{C}K$, using Theorem 6.1. The details are left to the reader.

We recall a definition of reduced suspension for c. s. s. complexes. The definition differs from the one given in [4], § 22; the first face and degeneracy operators play a special role instead of the last (see remark at the end of [4], § 1).

Definition 10.4. Let S be a reduced c. s. s. complex and ϕ its only 0-simplex. The *reduced suspension* of S is the c. s. s. complex $\bar{C}S$ obtained from the reduced cone CS (Definition 9.4) by identifying for every integer $n > 0$ and every simplex $\sigma \in S_n$

$$(0, \sigma) \text{ with } (n, \phi)$$

An immediate consequence of Theorem 9.5 then is

THEOREM 10.5. *Let S be a reduced c. s. s. complex. Then the c. s. s. homomorphism $\bar{h}\colon G\bar{C}S \to \bar{C}GS$ given by*

$$\bar{h}(\overline{k, \sigma}) = (k, \bar{\sigma})$$

is an isomorphism.

Finally we have

THEOREM 10.6. *Let the free c. s. s. group B be a loop complex of the reduced c. s. s. complex S under a twisting function t. Then $\bar{C}B$ is a loop complex of $\bar{C}S$.*

Proof. It follows from Proposition 4.4 that $gt\colon GS \to B$ is a loop homotopy equivalence and from Theorem 9.6 that $C(gt)\colon CGS \to CB$ is so. Consequently ([7], Theorem 5.3) their abelianizations $gt/[gt, gt]$ and

$C(gt)/[C(gt),C(gt)]$ are homotopy equivalences. Let $j\colon B \rightarrow CB$ be the c. s. s. monomorphism given by $j\sigma = (0,\sigma)$ for all $\sigma \in B$. Then

$$(CB/[CB,CB])/(j(B)/[j(B),j(B)]) \approx \bar{C}B/[\bar{C}B,\bar{C}B].$$

A similar relation holds for GS and it follows by application of the five lemmas ([2,], p. 16) that $\bar{C}(gt)/[\bar{C}(gt),\bar{C}(gt)]$ is also a homotopy equivalence. $\bar{C}GS$ and $\bar{C}B$ are connected and hence ([6], Theorem 6.1) the map $\bar{C}(gt)\colon \bar{C}GS \rightarrow \bar{C}B$ is a loop homotopy equivalence and the theorem now follows by application of Theorem 10.5 and Proposition 4.4.

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HARMONIC INTEGRALS ON FOLIATED MANIFOLDS.*

By Bruce L. Reinhart.

In the paper [5], we considered harmonic integrals on local product manifolds, that is, manifolds having two families of submanifolds in complementary dimensions, such that locally they look like the product of two euclidean spaces. The metric was assumed to be such that this local product could be taken in the sense of Riemannian manifolds. The results obtained were such as to suggest that analogous theorems could be proved if we assumed only one family of submanifolds, with a suitable choice of metric. This is indeed the case. We shall show in § 4 that on compact manifolds, the cohomology of base-like differential forms (defined in § 2) is isomorphic to the harmonic space of a certain semi-definite Laplacian (defined in § 3). The metric is assumed to be bundle-like in the sense of [6]; that paper may be referred to for examples of foliated manifolds possessing such a metric. In § 5, we discuss the meaning of our harmonic integral theorem for these examples.

1. Definitions. By a manifold M, we mean a C^∞ differentiable manifold; topologically, it is a connected, orientable, separable, locally euclidean Hausdorff space. We shall assume given on M (of dimension n) a C^∞ completely integrable q form Θ, that is, a locally decomposable, non-zero q form such that locally $d\Theta$ is a multiple of Θ [6]. A manifold with such a form will be called a foliated manifold. We shall also assume given a locally decomposable p $(=n-q)$ form Ω such that $\Omega \wedge \Theta$ is never zero. (The existence of such an Ω follows from the existence of Θ [6]. Since Ω does not occur in the final theorem, the fact that it is not unique does not matter). Given such a Θ, we can find about each point a coordinate neighborhood with coordinates $(x^1,\cdots,x^p,y^1,\cdots,y^q)$ such that

(i) $|x^i| \leqq 1$, $|y^\alpha| \leqq 1$,

(ii) The integral manifolds of Θ are given locally by $y^1=c^1,\cdots,$ $y^q=c^q$ for constants c^α satisfying $|c^\alpha| \leqq 1$.

* Received July 16, 1958.

(Here and hereafter, Latin indices run from 1 to p, and Greek indices from 1 to q.) Such a coordinate neighborhood will be called flat, while each of the slices given by a set of equations $y^\alpha = c^\alpha$ will be called a plaque. If U is a flat neighborhood, the quotient space of U by its plaques will be called the local base and be denoted by U_y. The natural projection $\pi\colon U \to U_y$ will be called the local projection.

Since Ω is locally decomposable, we may assume [6] that there exist in U differential forms ω^i and vectors v_α such that:

(i) $\Omega = \omega^1 \wedge \cdots \wedge \omega^p$ in U,

(ii) $\{\omega^1, \cdots, \omega^p, dy^1, \cdots, dy^q\}$ and $\{\partial/\partial x^1, \cdots, \partial/\partial x^p, v_1, \cdots, v_q\}$ are dual bases for the cotangent and tangent spaces respectively at each point of U. Hence, $\omega^i = dx^i + \sum a_\alpha{}^i dy^\alpha$ and $v_\alpha = \partial/\partial y^\alpha + \sum b_\alpha{}^i \partial/\partial dx^i$.

Throughout this paper, all local expressions for differential forms and vectors will be taken with respect to these bases.

2. Base-like cohomology. Our aim in this paragraph is to compute the special properties of the exterior derivative which hold for a foliated manifold, and to show how they may be applied to define a notion of base-like cohomology groups. These properties arise from the decomposition of differential forms into components in the following way: Any m form may be expressed locally as

$$\sum_{\substack{i_1 < \cdots < i_r \\ \alpha_1 < \cdots < \alpha_s}} \sum_{r+s=m} \phi_{i_1 \cdots i_r \alpha_1 \cdots \alpha_s}(x, y)\, \omega^{i_1} \wedge \cdots \wedge \omega^{i_r} \wedge dy^{\alpha_1} \wedge \cdots \wedge dy^{\alpha_s}.$$

We then define $\Pi_{r,s}\phi$ to be the sum of all those terms having a fixed r and s. Since under change of flat coordinate system, $\{\{dy^\alpha\}\}$ goes into $\{\{dy^{*\alpha}\}\}$ and $\{\{\omega^i\}\}$ goes into $\{\{\omega^{*i}\}\}$, this concept is independent of the choice of coordinate system. Here by $\{\{a^i\}\}$ we mean the vector space generated by the elements a^i. $\Pi_{r,s}\phi$ is called the component of type (r, s) of ϕ. (This notion is defined intrinsically in [4]). The type decomposition of differential forms induces a type decomposition of the exterior derivative d by the rule $(\Pi_{t,u}d)\phi = \sum_{r,s} \Pi_{r+t,s+u} d \Pi_{r,s}\phi$. Let $\Pi_{1,0}d = d'$ and $\Pi_{0,1}d = d''$. In general, there will be a component $\Pi_{-1,2}d$; since we are interested only in forms of type $(0, s)$, we shall not introduce a notation for this component.

PROPOSITION 1. *If ϕ is of type $(0, s)$, then $d\phi = d'\phi + d''\phi$. Moreover, $d'\phi = 0$ if and only if ϕ depends only upon y, in the sense that locally $\phi = \sum \phi_{\alpha_1 \cdots \alpha_s}(y)\, dy^{\alpha_1} \wedge \cdots \wedge dy^{\alpha_s}$.*

Proof. Any ϕ of type $(0,s)$ has the local expression

$$\phi = \sum \phi_{\alpha_1 \cdots \alpha_s}(x,y)\, dy^{\alpha_1} \wedge \cdots \wedge dy^{\alpha_s}.$$

Hence

$$d\phi = \sum (\partial\phi_{\alpha_1 \cdots \alpha_s}/\partial x^i)\, \omega^i \wedge dy^{\alpha_1} \wedge \cdots \wedge dy^{\alpha_s}$$
$$+ \sum (\partial\phi_{\alpha_1 \cdots \alpha_s}/\partial y^\alpha - a_\alpha{}^i\, \partial\phi_{\alpha_1 \cdots \alpha_s}/\partial x^i)\, dy^\alpha \wedge dy^{\alpha_1} \wedge \cdots \wedge dy^{\alpha_s}$$

where ω^i and $a_\alpha{}^i$ have the same meaning as in § 1, $\omega^i = dx^i + a_\alpha{}^i dy^\alpha$. The desired type decomposition is immediately obvious. Furthermore, $d'\phi = 0$ if and only if for every set of indices $\{\alpha_1, \cdots, \alpha_s, i\}$, $\partial\phi_{\alpha_1 \cdots \alpha_s}/\partial x^i = 0$. Since we assume that all forms considered are of class C^∞, this proves the proposition.

Definition. A form of type $(0,s)$ which is annihilated by d' will be called a base-like form.

Let A'' be the vector space of global differential forms which satisfy the conditions of Proposition 1; $A'' = \sum A''^s$, where A''^s consists of forms of type $(0,s)$. Restricted to A'', $d''^2 = d^2 = 0$, so we may consider the cohomology of A'' under d''. Call the groups H''^s so defined the base-like cohomology groups of M. We shall show that in the situation of a bundle-like metric on a compact manifold, these groups are finite dimensional and satisfy the usual duality relations.

Given any orientable differentiable manifold, we can always choose coordinate systems so that the Jacobians of the coordinate changes in overlapping neighborhoods are identically equal to 1. A generalization of this fact is essential in what follows.

PROPOSITION 2. *Given any foliated manifold covered by flat neighborhoods U_ν with coordinates $(x_\nu{}^1, \cdots, x_\nu{}^p, y_\nu{}^1, \cdots, y_\nu{}^q)$, we can rechoose coordinates so that the partial Jacobians $J_{\nu\mu} = (\partial x_\nu{}^i/\partial x_\mu{}^j)$ are identically* 1, *provided that they are all positive.*

Proof. Consider the sheaf of germs of positive real valued functions, defined by exponentiating the sheaf of germs of real valued functions. This is a fine sheaf since the exponential map is an isomorphism. $\{J_{\nu\mu}\}$ is a 1-cocycle with coefficients in this sheaf; hence, we can choose functions J_ν in U_ν such that $J_{\nu\mu} = J_\mu/J_\nu$. Introduce new coordinates by the formulas:

$$X_\nu{}^1 = \int_0^{x_\nu} J_\nu(t, x_\nu{}^2, \cdots, x_\nu{}^p, y_\nu{}^1, \cdots, y_\nu{}^q)\, dt,$$
$$X_\nu{}^i = x_\nu{}^i, \qquad i = 2, \cdots, p,$$

$$Y_\nu{}^\alpha = y_\nu{}^\alpha, \qquad \alpha = 1, \cdots, q.$$

Then $(\partial X_\nu{}^i / \partial X_\mu{}^j = J_\nu \cdot J_{\nu\mu} \cdot 1/J_\mu \equiv 1.$

3. The Laplacian for base-like forms. We wish to define a Laplacian operator which will map the set of base-like forms into itself; this may be thought of as a Laplacian in the variables y. For this to be possible, we need to assume the existence of a bundle-like metric.

Definition. A Riemannian metric is bundle-like if and only if it is representable in each flat neighborhood by an expression of the form

$$ds^2 = \sum g_{ij}(x, y) \omega^i \omega^j + \sum g_{\alpha\beta}(y) dy^\alpha dy^\beta.$$

In this definition, the essential restriction is that on the coefficients $g_{\alpha\beta}$; if these were allowed to depend on all variables, a metric could always be found satisfying the definition [5].

The tensor $\sum g_{\alpha\beta}(y) dy^\alpha dy^\beta$ defines a duality operation on the base-like forms by the formula

$$({*''}\phi)_{\alpha_1 \cdots \alpha_{q-s}} = \sum_{\beta_1 < \cdots < \beta_s} \delta^{1 \cdots q}{}_{\beta_1 \cdots \beta_s \alpha_1 \cdots \alpha_{q-s}}$$
$$(\det(g_{\alpha\beta}))^{\frac{1}{2}} g^{\beta_1 \nu_1} \cdots g^{\beta_s \nu_s} \phi_{\nu_1 \cdots \nu_s}$$

Then, in analogy with the ordinary case, we define

$$\delta'' \phi = (-1)^{qs+q+1} {*''} d'' {*''} \phi$$

where ϕ is a base-like form of degree s, and

$$\Delta'' = d'' \delta'' + \delta'' d''.$$

We wish to introduce a measure on M in such a way that Δ'' is self-adjoint in the Hilbert space of square integrable base-like forms. For this, Proposition 2 is the chief tool. Let $dV'' = (\det(g_{\alpha\beta}))^{\frac{1}{2}} dy^1 \wedge \cdots \wedge dy^q$ be the base-like volume element. In the coordinate neighborhood U_ν, we may consider the form $dx_\nu{}^1 \wedge \cdots \wedge dx_\nu{}^p$. Under coordinate changes, if we choose the neighborhoods as in Proposition 2, this goes into $dx_\mu{}^1 \wedge \cdots \wedge dx_\mu{}^p +$ terms involving the $dy_\mu{}^\alpha$. Taking the exterior product with dV'', the latter terms drop out and we find that $dV = dV'' \wedge dx^1 \wedge \cdots \wedge dx^p$ is a well-defined differential form giving a volume element on M. This in turn defines an inner product on base-like forms by the formula

$$(\phi, \psi) = \int_M \phi \wedge {*''}\psi \wedge dx^1 \wedge \cdots \wedge dx^p.$$

We next need to derive the formulas covering adjointness on a compact manifold. For this purpose, we recall that d'' restricted to base-like forms is the same as d restricted to base-like forms. Hence, for ϕ a base-like form and ψ a base-like s form, both of C^∞, we have

$$0=\int_M d(\phi\wedge *''\psi\wedge dx^1\cdots dx^p)=\int_M d''(\phi\wedge *''\psi)\wedge dx^1\wedge\cdots\wedge dx^p$$
$$=\int_M d''\phi\wedge *''\psi\wedge dx^1\wedge\cdots\wedge dx^p-\int_M \phi\wedge *''\delta''\psi\wedge dx^1\wedge\cdots\wedge dx^p,$$

which means $(d''\phi,\psi)=(\phi,\delta''\psi)$ for all base-like forms of arbitrary degree. It is clear that then $(\Delta''\phi,\psi)=(\phi,\Delta''\psi)$ for all base-like forms. If we complete in the given norm to get a Hilbert space, then the technique of Gaffney [2] enables us to show that if we take the closures of d'' and δ'' and denote them by the same symbols, then $\Delta''=d''\delta''+\delta''d''$ is a positive self-adjoint operator.

4. Green's operator. In this paragraph, we shall assume that M is compact and show how the techniques of [5] can be modified to show the existence on the Hilbert space of base-like differential forms of a bounded self-adjoint operator G'' such the $\Delta''G''\phi=\phi-H''\phi$ and $G''H''\phi=0$, where H'' is the projection of ϕ onto the kernel of the closed operator Δ''. The concept of coherent sheaf introduced there is not needed here; we merely restrict our attention to base-like forms. All symbols $'$ of that paper become $''$ here, and the role of the variables x and y is interchanged. In the paper [5], much use is made of the fact that a torsionless almost product metric induces a product measure on local product neighborhoods; moreover, these neighborhoods may be chosen with orthonormal coordinates. In this paper, we use instead a finite covering of M by neighborhoods such that the Jacobian in x is identically 1. At any point, we can choose a subneighborhood of one of the given neighborhoods which is orthonormal in y and has all $y^\alpha=0$ at the given point; this can be done without changing the variables x. Then the form dV'' induces a product measure on each such neighborhood, and the subneighborhoods may be used in the proof of the (purely local) differentiability lemma, the only place where the orthonormal coordinate system is used [5, Theorem 6.2]. We thus derive the main theorem of this paper. Because of the restriction in Proposition 2, we need the concept of orientable foliation.

Definition. A foliation will be said to be orientable if a covering by flat

coordinate neighborhoods can be chosen so that $(\partial x_\nu^i/\partial x_\mu^j) > 0$ in each intersection of two of them.

THEOREM. *Let M be a compact manifold with orientable foliation and bundle-like metric. Then there is defined on the Hilbert space of base-like forms on M a bounded symmetric operator G'' such that*

$$\Delta'' G''\phi = G''\Delta''\phi = \phi - H''\phi,$$
$$G''H''\phi = H''G''\phi = 0,$$
$$d''G''\phi = G''d''\phi, \qquad \delta''G''\phi = G''\delta''\phi.$$

Moreover, the kernel of Δ'' is finite dimensional.

Proof. The theorem is proved by the techniques used in proving Theorem 6.3 and Lemma 6.4 of [5], including therefore all the preceding parts of § 6. The modifications mentioned above are understood to be made throughout.

Let $b^s(y)$ be the dimension of the cohomology group of base-like forms of degree s on M.

COROLLARY. *$b^s(y)$ is finite, and $b^s(y) = b^{q-s}(y)$.*

Proof. This follows by standard techniques. Indeed, $d''\phi = 0$ implies that $\phi = d''\delta''G''\phi + H''\phi$, so is cohomologous to its harmonic part. If $H''\phi = d''\psi$, then $(H''\phi, H''\phi) = (H''\phi, d''\psi) = (\delta''H''\phi, \psi) = 0$; hence each cohomology class contains just one harmonic form. Thus, the cohomology groups are additively isomorphic to the kernel of Δ''. The finiteness of $b^s(y)$ follows immediately, while the equation $*''H''\phi = H''*''\phi$ implies the desired duality property.

5. Examples. We shall conclude the paper by showing how the theory just developed applies to some examples of foliated manifolds with bundle-like metrics. For notation, and proof that these examples satisfy the definition, see [6].

The first case to be considered is that of a fibre space (M, π, B). Here M is foliated by the subsets $\pi^{-1}(b)$ for $b \in B$; assume these fibres are connected.

PROPOSITION 3. *ϕ is base-like on M if and only if $\phi = \pi^*(\phi_0)$ for some ϕ_0 on B, where π^* is the map on differential forms induced by the projection π.*

Proof. Suppose first that $\phi = \pi^*(\phi_0)$, where ϕ_0 is an s form. By the local triviality, we see that ϕ must be of type $(0, s)$. Moreover, its value is independent of the point along a given fibre; hence, locally the coefficients

depend only upon y. Conversely, let $f\colon U \to M$ be a local cross-section, where U is a contractible euclidean neighborhood in B. Then $\phi_0 = f^*(\phi \mid f(U))$ is a form on U independent of f. Moreover, it clearly has the property that $\pi^*(\phi_0) = \phi$. ϕ_0 is clearly independent of the choice of flat coordinate system in M and corresponding coordinate system in B. This proves the proposition.

Interpreting the theorem for the case of fibre spaces, we see that it is essentially nothing but the usual Hodges' theorem for B; the only difference lies in that all computations may be carried out in M without referring to B explicitly.

The next example of interest is that of the foliation of M by orbits of a group of isometries H, under the assumption that all of these orbits are of the same dimension. If X is an element of the Lie algebra of H, it induces a vector field X_* on M which is everywhere tangent to the orbits of H; in fact, the set of all X_* generates the tangent space to the orbit thropgh each point. Let $i(X_*)$ denote the operation of contraction by X_*. Then Koszul [3] has defined the notion of *forme basique*; it is a differential form on M which satisfies the two conditions:

(i) $i(X_*)(\phi) = 0$ for all X in the Lie algebra of H.

(ii) $h_*(\phi) = \phi$ for all $h \in H$.

PROPOSITION 4. *Let the connected Lie group H act on M as a group of isometries with all orbits of the same dimension. Then ϕ is base-like if and only if* (i) *and* (ii) *hold.*

Proof. Suppose (i) and (ii) hold, and look at ϕ in a flat coordinate system. Since $\{X_*\}$ generates the same space as $\{\partial/x^i\}$, (i) implies that ϕ is expressible in terms of dy alone. By (ii), its value is independent of the point on the orbit, that is, of x. Thus, ϕ is base-like. Conversely, if ϕ is base-like, it clearly satisfies (i). Moreover, for h near the identity, (ii) may be proved by looking at a local coordinate system. To prove it for distant h, join them by a path, which may be covered by a finite number of translates of a suitable neighborhood of the identity. Since H is connected, the theorem is proved.

As our example, consider the unit tangent bundle M to a V-manifold B with isolated singularities; this is a sort of singular fibre space, in which the fibres are spheres except over the singular points, where they are spherical space forms. Let $(U_\alpha, G_\alpha, \eta_\alpha)$ be a local uniformizing structure in B; then a differential form ψ on $\tilde{U}_\alpha = \eta_\alpha(U_\alpha)$ is defined as a form on U which is invariant by G_α. It is clear that $\pi^*(\psi)$, defined on $\pi^{-1}(\tilde{U}_\alpha)$, is base-like

(by the same argument as in Proposition 3) and invariant by G_α. The proof of Proposition 3 may then be generalized to prove the following:

PROPOSITION 5. *Let M be the unit tangent bundle of the V-manifold B with isolated singularities. Then the base-like forms of M are precisely those induced from B by the projection mapping.*

It follows, just as in the fibre space case, that Hodge's theorem holds for B. This is a special case of the theorem of Baily [1] for arbitrary compact V-manifolds.

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Except when informality in referring to papers or books is called for by the context, the following form is preferred:

[3] O. K. Blank, "Zur Theorie des Untermengenraumes der abstrakten Leermenge," *Bulletin de la Societé Philharmonique de Zanzibar*, vol. 26 (1891), pp. 242-270.

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